

HOMEOMORPHISMS OF S^1 AND FACTORIZATION

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ABSTRACT. For each $n > 0$ there is a one complex parameter family of homeomorphisms of the circle consisting of linear fractional transformations ‘conjugated by $z \rightarrow z^n$ ’. We show that these families are free of relations, which determines the structure of ‘the group of homeomorphisms of finite type’. We also discuss a number of questions regarding factorization for more robust groups of homeomorphisms of the circle in terms of these basic building blocks, and the correspondence between smoothness properties of the homeomorphisms and decay properties of the parameters.

0. INTRODUCTION

In these notes we consider the question of whether it is possible to factor an orientation preserving homeomorphism of the circle (belonging to a given group) as a composition of ‘linear fractional transformations conjugated by $z \rightarrow z^n$ ’. What we mean by factorization depends on the group of homeomorphisms we are considering. In the introduction, as in the text, we will start with the simplest classes of homeomorphisms and build up. For algebraic homeomorphisms, factorization is to be understood in an algebraic way. For less regular homeomorphisms factorization involves limits and ordering, and in particular is highly asymmetric with respect to inversion. This very quickly leads to (very attractive) analytic questions which are beyond my ability to resolve.

0.1. Homeomorphisms of Finite Type. Given a positive integer n and $w_n \in \Delta := \{w \in \mathbb{C} : |w| < 1\}$, define a function $\phi_n : S^1 \rightarrow S^1$ by

$$(0.1) \quad \phi_n(w_n; z) := z \frac{(1 + \bar{w}_n z^{-n})^{1/n}}{(1 + w_n z^n)^{1/n}}, \quad |z| = 1$$

It is straightforward to check that $\phi_n \in \text{Diff}(S^1)$, the group of orientation preserving diffeomorphisms of S^1 , and $\phi_n^{-1}(z) = \phi_n(-w_n; z)$. We will refer to the subgroup generated by the ϕ_n , as w_n and n vary, as the group of homeomorphisms (or diffeomorphisms) of finite type.

Theorem 0.1. (a) *If n and m are relatively prime, then the set of diffeomorphisms $\{\phi_n(w_n), \phi_m(w_m) : w_n, w_m \in \Delta\}$ generates a dense subgroup of $\text{Diff}(S^1)$.*

(b) *If σ is a homeomorphism of finite type, then σ has a unique factorization*

$$\sigma = \lambda \circ \phi_{i_n}(w_{i_n}) \circ \dots \circ \phi_{i_1}(w_{i_1})$$

where $\lambda \in S^1$ is a rotation, $w_{i_j} \in \Delta \setminus \{0\}$, $j = 1, \dots, n$, and $i_j \neq i_{j+1}$, $j = 1, \dots, n-1$, for some n .

This is proven in Section 2.

Remark 1. ϕ_n satisfies the reality condition $\phi_n(z^*) = \phi_n(z)^*$ (where $z \rightarrow z^*$ is complex conjugation) if and only if w_n is real. For the subgroup of homeomorphisms of finite type which satisfy this reality condition and fix the points $\pm 1 \in S^1$ (i.e. the group of finite type homeomorphisms of an open string), the theorem implies that this subgroup is isomorphic to a countable free product $\mathbb{R} * \mathbb{R} * \mathbb{R} * \dots$

0.2. Algebraic Homeomorphisms. The subgroup of homeomorphisms of finite type is possibly comparable to the subgroup of the loop group of a compact Lie group consisting of finite type loops, i.e. loops with finite Laurent expansion (see e.g. Propositions 3.5.3, 5.2.5 and 8.8.1 of [26], and Theorem 1.2 of [28]). What we are missing is a direct characterization for homeomorphisms of finite type.

Question 1. *A homeomorphism $\sigma = \sigma(z)$ of finite type is an algebraic function, i.e. satisfies a polynomial equation $p(z, \sigma(z)) = 0$. Is the converse true?*

If the converse is false (and it is quite possible that I have overlooked a simple counterexample), then factorization simply fails for the group of algebraic homeomorphisms. In this case one might think of the group of algebraic homeomorphisms as more akin to rational loops in a compact Lie group (see e.g. [7]).

0.3. Diffeomorphisms. Theorem 0.1 is a unique factorization result for homeomorphisms of finite type. In the rest of the paper we are interested in factorization for more robust groups of homeomorphisms of the circle, and for semigroups of increasing functions on the line. This involves taking limits.

Fix a permutation of the natural numbers, $p : \mathbb{N} \rightarrow \mathbb{N} : n \rightarrow n'$. Given a sequence $w = (w_n) \in \prod_{n=1}^{\infty} \Delta$, define

$$(0.2) \quad \sigma_N = \phi_{N'} \circ \dots \circ \phi_{1'} \in \text{Diff}(S^1)$$

More explicitly (in particular to emphasize the dependence on parameters)

$$(0.3) \quad \sigma_N(p, w; z) = z \prod_{n=1}^N \frac{(1 + \bar{w}_{n'} \sigma_{n-1}(z)^{-n'})^{1/n'}}{(1 + w_{n'} \sigma_{n-1}(z)^{n'})^{1/n'}}, \quad |z| = 1$$

If $\sum_{n>0} \frac{1}{n} |w_n| < \infty$ (a condition which does not depend on p), then this product converges absolutely as $N \rightarrow \infty$, and hence the limit is a degree one surjective continuous function $S^1 \rightarrow S^1$. It is not clear when this limit is an invertible function, hence a homeomorphism of S^1 . We first consider a kind of core result, where this invertibility question is not much of an issue.

Theorem 0.2. *Fix a permutation p as above. For $s = 1, 2, \dots$, if $w \in \prod_{n=1}^{\infty} \Delta$ and $\sum_{n>0} n^{s-1} |w_n| < \infty$, then for $z \in S^1$ the limit*

$$\sigma(p, w; z) = z \prod_{n=1}^{\infty} \frac{(1 + \bar{w}_{n'} \sigma_n(z)^{-n'})^{1/n'}}{(1 + w_{n'} \sigma_n(z)^{n'})^{1/n'}}$$

exists and $\sigma(z) = \sigma(p, w; z)$ is a C^s homeomorphism of S^1 .

A key point is, because we assume $s \geq 1$, the inverse function theorem implies that the inverse of σ has the same degree of smoothness as σ . In general

$$\sigma_N^{-1} = \phi_{1'}(-w_{1'}) \circ \phi_{2'}(-w_{2'}) \circ \dots \circ \phi_{N'}(-w_{N'})$$

This does not have an expression analogous to (0.3) which is as useful in understanding convergence. In this smooth context, the inverse function theorem enables us to overcome this inversion asymmetry.

This leads to the basic stumbling block of these notes.

Question 2. *Is the map*

$$S^1 \times \left(c^\infty \cap \prod_{n=1}^{\infty} \Delta \right) \rightarrow \text{Diff}(S^1) : (\lambda; w) \rightarrow \lambda \sigma(p, w; z)$$

a bijection, where c^∞ is the Frechet space of rapidly decreasing sequences?

Because $\text{Diff}(S^1)$ is a Frechet (and not a Banach) Lie group, one cannot use the inverse function theorem to linearize the question.

At one time I wholeheartedly believed this to be true, and I thought that the inverse could be described in a somewhat constructive way. The naive idea is roughly the following. Given $\phi = \sigma_N(p, w)$, to recover $w_{1'}$, we try to minimize a height function $w \rightarrow H(\phi \circ \phi_{1'}(-w))$. This is analogous to finding a Fourier coefficient by solving a distance minimization problem, with the difference that we must solve these problems in an ordering prescribed by p . The challenge is to find the right kind of height function, which will need to have some fairly miraculous properties. The obvious candidate is $w \rightarrow -\log(a(\phi \circ \phi_{1'}(-w)))$, where $\phi = lmau$ is the triangular factorization (or conformal welding) for ϕ . Sadly, this does not appear to work. Our consolation is that this does lead to other interesting questions.

All that follows is perhaps of limited interest if the answer to Question 2 is negative.

0.4. Less Regular Homeomorphisms. To go further, one needs a sharper criterion for the limit in (0.3) to be invertible. There appears to be a very attractive sufficient condition.

Conjecture 1. *Fix the permutation p as above. If $w \in l^2$, then $\sigma(p, w) := \lim_{N \rightarrow \infty} \sigma_N \in \text{Homeo}(S^1)$.*

The l^2 condition is motivated by a glance at the derivative for σ (see (0.7) below).

We are interested in (decay) conditions on the parameters (w_n) which are equivalent to asserting that the corresponding homeomorphisms (augmented with rotations) form a group (defined by some smoothness condition). The following is probably a fantasy, but I cannot rule it out.

Question 3. *Fix the permutation p as above.*

(a) *If $w \in l^2$, is $\sigma(p, w) \in AC(S^1)$, the group of homeomorphisms which fix the Lebesgue measure class?*

(b) *If (a) is true, is the map*

$$S^1 \times \left(l^2 \cap \prod_{n=1}^{\infty} \Delta \right) \rightarrow AC(S^1) : (\lambda, w) \rightarrow \lambda \circ \sigma(p, w)$$

a bijection?

In addition to the form of the derivative, one vague supporting bit of evidence is that $AC(S^1)$ is the strong operator closure of $\text{Diff}(S^1)$ in its natural unitary representation on half densities, hence it is a group which has a ‘Euclidean description’; see Appendix B. One vague opposing bit of evidence is an analogy (of unknown reliability) with the theory of Verblunsky coefficients, see Appendix A and Szego’s Theorem 8.1 of [30].

A safer conjecture is that there exist correspondences of the form

$$(0.4) \quad S^1 \times \left(w^s \cap \prod_{n=1}^{\infty} \Delta \right) \rightarrow W^{s+1, L^2} \text{Homeo}(S^1)$$

for $s \geq 1/2$, where w^s denotes the space of sequences w satisfying $\sum n^s |w_n|^2 < \infty$, and the target is the group of homeomorphisms which are Sobolev of order $1 + s$ in the L^2 sense. Unfortunately, even for integer s , it seems far more difficult to find Sobolev estimates similar to the C^s estimates of the previous subsection. The critical case $s = 1/2$ is far and away the most interesting case; in this critical context one must understand “the group of homeomorphisms of order $1 + 1/2$ ” in a sense explained in Appendix B. In this critical case the completion is the strong operator closure of $\text{Diff}(S^1)$ in a Fock space representation, and hence this critical group also has a ‘Euclidean description’; see Appendix B. In this case the Verblunsky analogy works in favor (0.4) (see the Strong Szego Theorem 8.5 of [30]), and there is a heuristic argument (supported by some numerical evidence) which has worked in other contexts.

0.5. Almost Sure Type Questions. In other contexts (e.g. compact Lie groups, and their associated loop groups), similar factorizations are useful because Toeplitz determinants, Haar type measures, and (conjecturally) related Poisson structures factor in these coordinates (see [24],[25]).

There are a number of known interesting probability measures on $\text{Homeo}(S^1)$, with diverse origins (e.g. see [3], [17], [18], [23], and references). One example is related to Werner’s work on conformally invariant measures on self-avoiding loops on Riemann surfaces. In this case it is of interest to consider the welding map from topologically nontrivial loops in the punctured plane to homeomorphisms of S^1 ,

$$(0.5) \quad W : \text{Loop}^1(\mathbb{C} \setminus \{0\}) \rightarrow \text{Homeo}(S^1) : \gamma \rightarrow \sigma(\gamma) := \phi_-^{-1} \circ \phi_+$$

where ϕ_{\pm} are appropriately normalized uniformizations for the regions interior and exterior to γ , respectively, and the image of Werner’s measure with respect to this map (see the Introduction to [6] for more detail, and references). In subsection 4.3 we will speculate that the inverse to the product map in part (b) of Question 2 might have an extension to a map

$$(0.6) \quad \text{Loop}^1(\mathbb{C} \setminus \{0\}) \rightarrow \prod_{n=1}^{\infty} \Delta : \gamma \rightarrow w$$

at least in an almost sure sense. I have not found any (e.g. Poisson) geometrical structure which suggests that the image of Werner’s measure, or any other natural measure, is a product in terms of the conjectural parameters (w_n) . However all of the measures alluded to above are related to the critical exponent $s = 1/2$; at least in a heuristic sense the group of $W^{1+1/2, L^2}$ homeomorphisms, understood in the sense of Appendix B, is analogous to a Cameron-Martin type space for these measures. In any event it is interesting to reconsider the invertibility question of the previous subsection in a probabilistic background.

Consider a probability measure on $\prod_{n=1}^{\infty} \Delta$ of the form

$$\prod_{n=1}^{\infty} \frac{1}{3_n} (1 - |w_n|^2)^{a(n)} |dw_n|$$

where $a(n)/n \rightarrow \alpha$ as $n \rightarrow \infty$.

Conjecture 2. *Suppose p is the trivial permutation. There exists α_0 such that for $\alpha < \alpha_0$, with probability one, $\sigma(p, w)$ is a homeomorphism of S^1 having a unique conformal welding (see Theorem 1.1 in Subsection 1.3).*

The motivating measures mentioned above are expected to come in families (in the case of Werner's work, this conjectural family, parameterized by central charge, is described in [17]), and the critical α_0 should be related to a condition on the central charge.

0.6. Increasing Functions on the Line. We continue to fix a permutation p of \mathbb{N} . We now propose to simply ignore the invertibility question for the limit of the σ_N , and attempt to imitate the theory of Verblunsky coefficients.

Suppose that $w \in \prod_{n=1}^{\infty} \Delta$. Write

$$\sigma_N(p, w; e^{i\theta}) = e^{i\Sigma_N(p, w; \theta)}$$

where the lift Σ_N is a homeomorphism of \mathbb{R} satisfying

$$\Sigma_N(\theta + 2\pi) = \Sigma_N(\theta) + 2\pi;$$

Σ_N is uniquely determined modulo $2\pi\mathbb{Z}$. Analogous to (0.3),

$$\Sigma_N(\theta) = \theta - 2 \sum_{n=1}^N \frac{1}{n'} \Theta(1 + w_{n'} \sigma_{n-1}(e^{i\theta})^{n'}) \pmod{2\pi\mathbb{Z}}$$

where $-\frac{\pi}{2} < \Theta < \frac{\pi}{2}$ is the polar angle. By the chain rule

$$(0.7) \quad d\Sigma_N(\theta) = \left(\prod_{n=1}^N \frac{1 - |w_{n'}|^2}{|1 + w_{n'} \sigma_{n-1}(z)^{n'}|^2} \right) \frac{d\theta}{2\pi}$$

This differential is independent of the choice of lift Σ_N , and we can interpret it as a probability measure on S^1 .

Question 4. *Fix a permutation p as above.*

(a) *Given $w \in \prod_{n=1}^{\infty} \Delta$, $\frac{1}{2\pi} d\Sigma_N$ has weak* limits in $\text{Prob}(S^1)$. There is a unique limit when $\sum \frac{1}{n} |w_n| < \infty$. Is there always a unique limit?*

(b) *Assuming that the limit in (a) is always unique, there is a map*

$$\prod_{n=1}^{\infty} \Delta \rightarrow \text{Prob}(S^1)$$

Is this injective, and does the image consist of measures having infinite support?

The Verblunsky analogue of the map in (b) has a finite algebraic character, with respect to the Fourier coefficients for a measure on S^1 ; in particular the Verblunsky map is relatively easy to invert. It is an amazing fact that the Verblunsky analogue of the map in (b) has an extension to a map

$$(0.8) \quad \prod_{n=1}^{\infty} D \rightarrow \text{Prob}(S^1)$$

where D is the closed unit disk, and this extension induces a homeomorphism from a compactification of $\prod_{n=1}^{\infty} \Delta$ to $\text{Prob}(S^1)$ with its weak* topology. The purported map in (b) (assuming it exists) cannot possibly extend to a map as in (0.8). For the Verblunsky map as in (0.8), two sequences go to the same measure if they agree up to the first entry which is in S^1 , so that the map is relatively simple at infinity;

for the map in (b) (assuming it exists, e.g. for $p = \text{identity}$) it can be checked that there is lack of uniqueness of the *weak** limit, as in (a) of the question, for a sequence of $w_n \in S^1$, which wanders too erratically around the circle. Thus the limits we are considering are definitely less well-behaved “at infinity” than for the Verblunsky map.

0.7. Ordering of the Factors. An overarching question is whether there is anything special about the obvious ordering of factors, $p = \text{identity}$. As illustrated above, our tentative hypothesis is that for absolute type conditions, there is no need to restrict p , whereas for almost sure type questions, restrictions on p are important. There is no analogue of this ordering issue in the theory of Verblunsky coefficients. In the analogy with the theory of loop groups, there is a need for ordering, there are special orderings (related to factorization in the associated Weyl group), Toeplitz determinants factor, measures factor, etc, in the associated coordinates. But the special Kac-Moody algebra structure of the loop group setting does not appear here. The lack of this special algebraic structure may simply doom this whole project.

0.8. Plan of the Paper. The first section reviews some basic facts about the Virasoro algebra and group. We also review triangular factorization (i.e. conformal weldings), and present some examples; this needs to be developed much more extensively.

In Section 2 we introduce the subgroup of diffeomorphisms of S^1 of “finite type”. We will see that Theorem 0.1 can be restated in the following way: the group of diffeomorphisms of finite type is the amalgam (i.e. the free product modulo the rotation subgroup intersection) of the covering groups $PSU(1, 1)^{(n)}$, $n = 1, 2, \dots$, of $PSU(1, 1)$, the group of linear fractional transformations which stabilize S^1 . In Section 3 we prove Theorem 0.2. In Section 4 we discuss height functions (which at one time we thought would lead to a proof of Question 2). In Section 5 we discuss Question 1. In Section 6 we briefly discuss increasing functions on the line. In Appendix A we briefly discuss how the coefficients w_n are somewhat analogous to Verblunsky coefficients from the theory of orthogonal polynomials (There is the very real possibility that we are overplaying this analogy). In Appendix B we recall some basic smoothness conditions for homeomorphisms of S^1 , from a group theoretic point of view.

0.9. Basic Notation. Homeomorphisms of S^1 are assumed to be orientation preserving, unless stated otherwise. Given a homeomorphism ϕ of S^1 , there is a homeomorphism Φ of \mathbb{R} such that

$$\phi(e^{i\theta}) = e^{i\Phi(\theta)}$$

Φ satisfies

$$(0.9) \quad \Phi(\theta + 2\pi) = \Phi(\theta) + 2\pi$$

and is uniquely determined up to the addition of a multiple of 2π . The set of homeomorphisms Φ of \mathbb{R} satisfying (0.9) is a realization of the universal covering group

$$0 \rightarrow 2\pi\mathbb{Z} \rightarrow \widetilde{Homeo}(S^1) \rightarrow Homeo(S^1) \rightarrow 0$$

where Φ projects to ϕ .

We use $s \geq 0$ to denote order of smoothness, in various senses. If $s = k$, where $k = 0, 1, 2, \dots$, then C^s is the space of functions f on \mathbb{R} or S^1 such that f is k -times continuously differentiable. If $s = k + \alpha$, where $k = 0, 1, 2, \dots$ and $0 < \alpha < 1$, then C^s is the space of functions f on \mathbb{R} or S^1 such that f is k -times differentiable and $f^{(k)}$ satisfies a Holder condition of order α (there is a natural Banach space topology for this space, but this will not play a role). In most contexts s is restricted to be integral. $W^s = W^{s;L^2}$ is the space of functions f on \mathbb{R} or S^1 which are L^2 Sobolev of order s .

We let (n, m) denote the greatest common divisor of positive integers n, m .

1. BACKGROUND

1.1. The Virasoro Algebra. The group of diffeomorphisms of S^1 (or more generally, any compact manifold) is a Frechet Lie group. The Lie algebra of $Diff(S^1)$ can be identified with smooth real vector fields on S^1 , with the negative of the traditional differential geometric bracket (see [19]). The complexification of this Lie algebra has a universal central extension by \mathbb{C} . The complex Virasoro algebra is the universal central extension of the Lie subalgebra of complex trigonometric vector fields on the circle. As a vector space

$$Vir = \left(\sum_{n \in \mathbb{Z}} \mathbb{C}L_n \right) \oplus \mathbb{C}\kappa$$

where

$$L_n = ie^{in\theta} \frac{d}{d\theta} = -z^{n+1} \frac{d}{dz}$$

The bracket is determined by the relations

$$(1.1) \quad [L_n, L_m] = (m - n)L_{n+m} + \frac{1}{12}n(n^2 - 1)\delta(n + m)\kappa; \quad [L_n, \kappa] = 0$$

The Virasoro algebra has a triangular decomposition, in the technical sense of [20],

$$Vir = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}^+, \text{ where } \mathfrak{n}^\pm = \sum_{\pm n > 0} \mathbb{C}L_n \text{ and } \mathfrak{h} = \mathbb{C}L_0 \oplus \mathbb{C}\kappa$$

Remark 2. For most purposes of this paper, the reader can ignore the central extension. The embeddings below can be viewed simply as embeddings into vector fields of the circle, and so on. But for some purposes the extension is essential.

For each $n > 0$, there is a Lie algebra embedding

$$di_n : sl(2, \mathbb{C}) \rightarrow Vir : \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \rightarrow f_n = -\frac{1}{n}L_{-n},$$

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \rightarrow h_n = \frac{2}{n}L_0 - \frac{1}{12n}(n^2 - 1)\kappa, \text{ and } \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \rightarrow e_n = \frac{1}{n}L_n$$

The restriction of di_n to $su(1, 1)$ is given by

$$(1.2) \quad di_n : \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \rightarrow ih_n, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightarrow \frac{1}{n}L_n - \frac{1}{n}L_{-n}, \text{ and } \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix} \rightarrow \frac{i}{n}L_n + \frac{i}{n}L_{-n}$$

1.2. The Virasoro Group. The group $Diff(S^1)$ has a universal central extension

$$0 \rightarrow \mathbb{Z} \times i\mathbb{R} \rightarrow \widehat{Diff(S^1)} \rightarrow Diff(S^1) \rightarrow 0$$

Bott observed that the group $\widehat{Diff(S^1)}$ can be realized in the following explicit way. As a manifold

$$\widehat{Diff(S^1)} = \widetilde{Diff(S^1)} \times i\mathbb{R}$$

In these coordinates the multiplication is given by

$$(\Phi; it) \cdot (\Psi; is) = (\Phi \circ \Psi; it + is + iC(\phi; \psi))$$

where C is the \mathbb{R} -valued cocycle given by

$$C(\phi; \psi) = \frac{1}{48\pi} \operatorname{Re} \int_{S^1} \log\left(\frac{\partial\phi}{\partial z} \circ \psi\right) d\left(\log\left(\frac{\partial\psi}{\partial z}\right)\right)$$

The corresponding Lie algebra is the real form of (the smooth completion of) Vir which as a vector space equals $\operatorname{vect}(S^1) \oplus i\mathbb{R}$ with the bracket given by (1.1).

Proof. One obtains the corresponding Lie algebra cocycle via

$$\begin{aligned} c(\vec{\xi}, \vec{\eta}) &= \frac{\partial}{\partial s \partial t} \Big|_{s=t=0} (C(e^{s\vec{\xi}}, e^{t\vec{\eta}}) - C(e^{t\vec{\eta}}, e^{s\vec{\xi}})) \\ &= \frac{i}{24\pi} \int_{S^1} \frac{\partial\xi}{\partial z} d\left(\frac{\partial\eta}{\partial z}\right) = \frac{i}{24\pi} \int_0^{2\pi} (\tilde{\eta}'''(\theta) + \tilde{\eta}'(\theta)) \tilde{\eta}(\theta) d\theta \end{aligned}$$

where $\vec{\xi} = \xi(z) \frac{d}{dz} = \tilde{\xi}(\theta) \frac{d}{d\theta}$. This gives the commutation relations in (1.1). \square

There is a Lie group embedding

$$i_n : PSU(1, 1)^{(n)} \rightarrow Diff(S^1)$$

corresponding to the Lie algebra embedding (1.2). We will write down this embedding in an explicit way in the next subsection. At the level of diffeomorphisms, it is understood geometrically as follows. The group of projective transformations of the Riemann sphere which map the circle to itself is $PSU(1, 1) \subset PSL(2, \mathbb{C})$, where

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \cdot z' = \frac{\bar{\eta} + \bar{\alpha}z'}{\alpha + \beta z'}$$

For $n \geq 1$ there is an n -fold covering map,

$$S^1 \rightarrow S^1 : z \rightarrow z' = z^n$$

The diffeomorphisms of z which cover the projective transformations of z' form a group $PSU(1, 1)^{(n)}$, which is a realization of the n -fold covering

$$(1.3) \quad 0 \rightarrow \mathbb{Z}_n \rightarrow PSU(1, 1)^{(n)} \rightarrow PSU(1, 1) \rightarrow 0$$

In [12] it is conjectured that every finite dimensional closed subgroup of $Homeo(S^1)$ is contained in a conjugate of one of the subgroups $PSU(1, 1)^{(n)}$.

1.3. Triangular factorization. To better understand $PSU(1, 1)^{(n)}$, and for later purposes, we recall the analogue of triangular factorization for homeomorphisms of S^1 , often referred to as conformal welding. Just as an invertible matrix may not have an LDU factorization, a general homeomorphism may not have a triangular factorization; unlike the matrix case, the existence of a triangular factorization does not imply that the factorization is unique. However for homeomorphisms which are quasimetric (a relatively mild regularity condition, with multiple characterizations - see Appendix B), the situation is completely straightforward.

Theorem 1.1. *Suppose that σ is a quasimetric homeomorphism of S^1 . Then*

$$\sigma = l \circ ma \circ u$$

where

$$u = z(1 + \sum_{n \geq 1} u_n z^n)$$

is a univalent holomorphic function in the unit disk Δ , with quasiconformal extension to \mathbb{C} , $m \in S^1$ is rotation, $0 < a \leq 1$ is a dilation, the mapping inverse to l ,

$$L(z) = z + \sum_{n \geq 0} b_n z^{-n}$$

is a univalent holomorphic function on the unit disk about infinity, Δ^* with quasiconformal extension to \mathbb{C} , and the compatibility condition

$$mau(S^1) = L(S^1)$$

holds. This factorization is unique.

For the state of the art, and especially for examples of homeomorphisms which are not weldings, and for weldings which are not unique, see [5].

Remark 3. If σ has triangular factorization $lmau$, then the triangular factorization of σ^{-1} is given by

$$u(\sigma^{-1})(z) = \frac{1}{L(\frac{1}{z^*})^*}; \quad l(\sigma^{-1})(z) = \frac{1}{U(\frac{1}{z^*})^*}; \quad m(\sigma^{-1}) = m(\sigma)^*; \quad a(\sigma^{-1}) = a(\sigma)$$

where L and U are inverse to l and u , respectively.

Suppose that $\phi \in PSU(1, 1)^{(n)}$, and suppose that ϕ covers $\pm \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} \in PSU(1, 1)$.

Corresponding to the matrix triangular factorization

$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \bar{\alpha} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \bar{\beta}\alpha^{-1} & 1 \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \begin{pmatrix} 1 & \alpha^{-1}\beta \\ 0 & 1 \end{pmatrix}$$

and setting $w_n = \alpha^{-1}\beta$, there is a heuristic factorization in the ‘complexification of $Diff(S^1)$ ’

$$\phi = \exp\left(\frac{-\bar{w}_n}{n} L_{-n}\right) \alpha^{\frac{2}{n} L_0 - \frac{1}{12n}(n^2-1)\kappa} \exp\left(\frac{w_n}{n} L_n\right)$$

(To make rigorous sense of this, one approach is to use formal completions, as in [23], but we will avoid this). At the level of diffeomorphisms, this can be understood rigorously as a triangular factorization, as in Theorem 1.1,

$$\phi = l(\phi) \circ ma(\phi) \circ u(\phi)$$

where

$$u(\phi)(z) = \frac{z}{(1 + w_n z^n)^{1/n}}, \quad ma(\phi) = \alpha^{-2/n}; \quad a(\phi) = (1 - |w_n|^2)^{1/n}$$

(where the root $\alpha^{1/n}$ is unambiguous because we are considering the n -fold covering of $PSU(1, 1)$), and

$$L(\phi)(z) = z(1 - \bar{w}_n z^{-n})^{1/n}$$

The composition is given explicitly by

$$(1.4) \quad \phi = \left(\frac{\bar{\alpha}}{\alpha} \right)^{1/n} \phi_n(w_n; z)$$

where again the n th root is unambiguous because we are considering the n -fold covering of $PSU(1, 1)$. The expression (1.4) implies part (a) of the following lemma. Part (b) is a straightforward calculation.

Lemma 1. (a) *Each element in $PSU(1, 1)^{(n)}$ can be written as*

$$Rot(\theta) \circ \phi_n(w_n; z)$$

for a uniquely determined rotation and $w_n \in \Delta$.

(b)

$$\phi_n(w_n) \circ \phi_n(w'_n) = e^{\frac{2i}{n}(1 + w_n \bar{w}'_n)} \phi_n(\phi_1(\bar{w}'_n; w_n); z)$$

Thus all of the subgroups $PSU(1, 1)^{(n)}$ have the rotation subgroup in common, and the transformations ϕ_n (parameterized by a disk) give a natural cross section for the projection from $PSU(1, 1)^{(n)}$ to the quotient modulo rotations.

1.4. More Examples of Triangular Factorization. It is an interesting question whether there is a procedure for calculating the triangular factorization for a composition $\phi_n \circ \dots \circ \phi_2 \circ \phi_1$. I only understand the most elementary cases.

Proposition 1. *Suppose that the triangular factorization of ϕ is known:*

$$\phi = l(\sigma)ma(\sigma)u(\sigma)$$

Then

$$L(\phi \circ \phi_1)(z) = L(\phi)(z) - (ma)(\phi)u(\phi)(\bar{w}_1), \quad |z| > 1;$$

$$(ma)(\phi \circ \phi_1) = (ma)(\phi)a(\phi_1)u(\phi)'(\bar{w}_1);$$

and

$$u(\phi \circ \phi_1)(z) = \frac{1}{u(\phi)'(\bar{w}_1)(1 - w_1 \bar{w}_1)} (u(\phi)\left(\frac{z + \bar{w}_1}{1 + w_1 z}\right) - u(\phi)(\bar{w}_1)); \quad |z| < 1$$

Remarks 1. (a) *This shows that if we consider a composition $\sigma_2 = \phi_2 \circ \phi_1$, it is not the case that the diagonal term factors, e.g. in general, $a(\sigma_2) \neq a(\phi_2)a(\phi_1)$. This is disappointing, because the analogue of this in the context of loop groups is true.*

(b) *There are also formulas for the triangular factorization of a composition of the form $\phi_1 \circ \phi$, because of Remark 3.*

Proof. We must check that for $|z| = 1$ (for our claimed formulas)

$$L(\phi \circ \phi_1) \circ \phi \circ \phi_1(z) = (ma)(\phi \circ \phi_1)u(\phi \circ \phi_1)(z)$$

The left hand side equals

$$L(\phi)(\phi(\phi_1(z))) - (ma)(\phi)u(\phi)(\bar{w}_1)$$

The right hand side equals

$$(ma)(\phi)a(w_1)^{-2}u(\phi)(\bar{w}_1)\frac{1}{u(\phi)'(\bar{w}_1)(1-w_1\bar{w}_1)}(u(\phi)(\phi_1(z)) - u(\phi)(\bar{w}_1))$$

$$(ma)(\phi)u(\phi)(\phi_1(z)) - (ma)(\phi)u(\phi)(\bar{w}_1)$$

The two sides are equal because

$$L(\phi) \circ \phi = (ma)(\phi)u(\phi)$$

□

Using this proposition we can also explicitly find the triangular factorization for a composition of the form $\phi \circ \phi_n$, when ϕ satisfies $\phi(z^n) = \phi(z)^n$, and the triangular factorization for ϕ is known. To see this, first note that at a heuristic level,

$$\begin{aligned}\phi \circ \phi_n(w_n) &= \phi \circ z^{1/n} \circ \phi_1(w_n) \circ z^n \\ &= z^{1/n} \circ (\phi \circ \phi_1(w_n)) \circ z^n\end{aligned}$$

Given a normalized univalent function $u = z(1 + \sum_{k=1}^{\infty} u_k z^k)$, there is a straightforward meaning attached to $z^{1/n} \circ u \circ z^n$:

$$z^{1/n} \circ u \circ z^n := z(1 + \sum_{k=1}^{\infty} u_k z^{nk})^{1/n}$$

In a similar way, if $L = z(1 + \sum_{k=0}^{\infty} b_k z^{-k})$,

$$z^{1/n} \circ L \circ z^n := z(1 + \sum_{k=0}^{\infty} b_k z^{-nk})^{1/n}$$

This is the origin of formula for $u(\phi_n)$.

Corollary 1. *Suppose that $\phi(z^n) = \phi(z)^n$, and suppose that the triangular factorization of ϕ is known:*

$$\phi = l(\sigma)ma(\sigma)u(\sigma)$$

Using the proposition we can find the triangular factorization for $\phi \circ \phi_1(w_n)$, and the triangular factorization for $\phi \circ \phi_n$ is given by

$$\begin{aligned}u(\phi \circ \phi_n) &= z^{1/n} \circ u(\phi \circ \phi_1(w_n)) \circ z^n \\ L(\phi \circ \phi_n) &= z^{1/n} \circ L(\phi \circ \phi_1(w_n)) \circ z^n \\ a(\phi \circ \phi_n) &= a(\phi \circ \phi_1(w_n))^{1/n}\end{aligned}$$

and

$$m(\phi \circ \phi_n) = m(\phi \circ \phi_1(w_n))^{1/n}$$

where the root must be resolved.

In particular we can find the triangular factorization of compositions of the form $\phi_{2k} \circ \phi_2$, $\phi_{3k} \circ \phi_3$, and so on, for $k > 1$.

On the other hand I do not know how to find the triangular factorization for something so seemingly simple as $\phi_3 \circ \phi_2$.

2. FINITE TYPE DIFFEOMORPHISMS AND FACTORIZATION

In this section we will prove Theorem 0.1. Because of Lemma 1, part (a) can be restated in the following way.

Theorem 2.1. *Suppose that n and m are relatively prime. Then the subgroup generated by $PSU(1, 1)^{(n)}$ and $PSU(1, 1)^{(m)}$ is dense in $Diff(S^1)$.*

The proof of this follows by a straightforward modification of the proof of Proposition 3.5.3 of [26] (which in turn relies on an argument that goes back to Cartan, used in his proof that a closed subgroup of a finite dimensional Lie group is a Lie subgroup).

Proof. Let G denote the C^∞ closure of the subgroup generated by $PSU(1, 1)^{(n)}$ and $PSU(1, 1)^{(m)}$ in $Diff(S^1)$. Let \mathfrak{g} denote the set of vector fields X such that the corresponding one parameter group is contained in G . In a standard way \mathfrak{g} is a vector space and a Lie algebra, using

$$\exp(t(X + Y)) = \lim_{n \rightarrow \infty} (\exp(tX/n) \circ \exp(tY/n))^n$$

and

$$\exp(t^2[X, Y]) = \lim_{n \rightarrow \infty} (\exp(tX/n) \circ \exp(tY/n) \circ \exp(-tX/n) \circ \exp(-tY/n))^{n^2}$$

It is obvious that \mathfrak{g} contains the Lie algebras of $PSU(1, 1)^{(n)}$ and $PSU(1, 1)^{(m)}$. We claim that this, together with $(n, m) = 1$, implies that \mathfrak{g} contains the Lie algebra of all trigonometric vector fields. To prove this, it suffices to show that if $(n, m) = 1$, then the Lie algebra generated by $L_{\pm n}$ and $L_{\pm m}$ is the entire Witt algebra. The repeated adjoint action of the $L_{\pm m}$ on L_n generates all L_{n+km} , $k \in \mathbb{Z}$; similarly the repeated adjoint action of the $L_{\pm n}$ on L_m generates all L_{m+ln} , $l \in \mathbb{Z}$. Now $(n, m) = 1$ implies that $\{km + ln : k, l \in \mathbb{Z}\} = \mathbb{Z}$. Thus the Lie algebra generated by $L_{\pm n}$ and $L_{\pm m}$ is the entire Witt algebra. This proves the claim.

It now follows that \mathfrak{g} is dense in smooth vector fields. Since \mathfrak{g} is C^∞ closed, \mathfrak{g} is the Lie algebra of all smooth vector fields. Thus all one parameter subgroups of $Diff(S^1)$ belong to G , and this implies $G = Diff(S^1)$. \square

Since the intersection of $PSU(1, 1)^{(n)}$ and $PSU(1, 1)^{(m)}$ is the group of rotations, part (b) of Theorem 0.1 can be restated in the following way.

Theorem 2.2. *The group of diffeomorphisms of finite type equals the amalgam of the subgroups $PSU(1, 1)^{(n)}$, $n = 1, 2, \dots$, i.e. it is the free product of these subgroups, modulo the obvious relations arising from the common intersection, $Rot(S^1)$.*

The maps ϕ_n , and their compositions, can be viewed as algebraic functions. We will use this point of view to derive invariants for compositions, especially the notion of degree.

2.1. Some Definitions. Suppose that Σ is a connected compact Riemann surface with nonempty boundary S (a disjoint union of circles). Let $\hat{\Sigma}$ denote the double, i.e.

$$\hat{\Sigma} = \Sigma^* \circ \Sigma$$

where Σ^* is the adjoint of Σ , the surface Σ with the orientation reversed, and the composition is sewing along the common boundary S . Let Θ denote the antiholomorphic involution (or reflection) fixing S . The basic example is the realization of the Riemann sphere as the double of the closed unit disk D , where $\Theta(z) = \frac{1}{z^*}$.

Definition 1. (a) A Riemann surface with reflection symmetry (or a surface with a real structure) is a connected compact Riemann surface $\hat{\Sigma}$ which is a double

$$\hat{\Sigma} = \Sigma^* \circ \Sigma$$

(b) A holomorphic map $f : \Sigma^* \circ \Sigma \rightarrow D^* \circ D$ is equivariant if it satisfies

$$f(\Theta(q)) = \frac{1}{f(q)^*}$$

and strictly equivariant if it additionally satisfies $f^{-1}(D) = \Sigma$.

Suppose that ϕ is an analytic homeomorphism of S^1 . Analyticity implies that there exists a reflection invariant domain Ω containing S^1 and an analytic continuation $\phi : \Omega \rightarrow \phi(\Omega)$ which is a conformal isomorphism. For $q \in \Omega$, this continuation will satisfy the equivariance condition in (b) of the Definition, and the continuation is strictly equivariant in the limited sense that $\Omega \cap \Delta$ will be mapped into Δ . In general there does not exist a maximal domain Ω .

Definition 2. A homeomorphism ϕ of S^1 is algebraic if there exists a polynomial $p(z, w)$ such that $p(z, \phi(z)) = 0$.

To say that ϕ is algebraic is equivalent to saying that ϕ has an analytic continuation to a multivalued function on a reflection invariant domain $\mathbb{P}^1 \setminus \{z_j, 1/z_j^* : 1 \leq j \leq n\}$ such that the singularities are algebraic (see e.g. [2], Theorem 4 of chapter 8).

The following is obvious.

Proposition 2. *The group of homeomorphisms of finite type is contained in the group of algebraic homeomorphisms.*

The interesting open question (assuming I have not overlooked some simple counterexample) is whether the converse is true.

Proposition 3. *Suppose that ϕ is an algebraic homeomorphism. Then there exist*

- (1) *a compact connected Riemann surface with reflection symmetry $\hat{\Sigma} = \Sigma^* \circ \Sigma$;*
 - (2) *strictly equivariant holomorphic maps $Z, W : \hat{\Sigma} \rightarrow D^* \circ D$;*
 - (3) *an irreducible polynomial p (of two variables over \mathbb{C}) such that $p(Z, W) = 0$;*
- and*
- (4) *a distinguished component of S , denoted S_1 , such that $Z; W : S_1 \rightarrow S^1$ are homeomorphisms, and $\phi = W \circ (Z|_{S_1})^{-1}$*

Proof. The Riemann surface defined by ϕ is the quotient of the universal covering of the punctured sphere $\mathbb{P}^1 \setminus \{z_j, 1/z_j^* : 1 \leq j \leq n\}$ by the group of automorphisms which fixes a single-valued lift of ϕ , where the z_j are the branch points for ϕ in Δ . There are other ways to describe this surface, such as by using germs of branches for analytic continuations of ϕ ; see e.g. chapter 8 of [2]. The punctured sphere is stable with respect to reflection, so this reflection symmetry lifts to the universal covering. Since ϕ is also reflection symmetric, this descends to a reflection symmetry for the Riemann surface defined by ϕ . Let \tilde{Z} denote the projection from this (incomplete) Riemann surface to the punctured sphere, and let \tilde{W} denote a single-valued lift of ϕ to the surface. Because of the reflection symmetry of the domain and ϕ , these are both strictly equivariant. These functions satisfy a polynomial equation $p(\tilde{Z}, \tilde{W}) = 0$, which we can suppose is irreducible. It is well-known that this implies

that the surface defined by ϕ can be extended to a compact Riemann surface $\hat{\Sigma}$ in a unique way so that \tilde{Z} and \tilde{W} extend to holomorphic maps Z and W (This is essentially the Riemann extension theorem, see Theorem 2 of [8]).

If ϕ is replaced by $\phi_1(w'_1) \circ \phi \circ \phi_1(w_1)$, then the Riemann surface remains the same, but

$$Z = \phi_1(w_1)^{-1} \circ Z_\phi \text{ and } W = \phi_1(w_1) \circ W_\phi$$

This illustrates that there is some nonuniqueness in our association of data to a homeomorphism of finite type. \square

Proposition 4. *Suppose that ϕ is an algebraic homeomorphism, as in the preceding Proposition 3. Then*

- (a) Z and W are homeomorphisms restricted to each connected component of S .
- (b) $\text{degree}(Z) = \text{degree}(W) = |\pi_0(S)|$ (the number of components of S), and this is also the degree of p , the essentially unique irreducible polynomial satisfying $p(Z, W) = 0$ in each of the variables Z and W .

Proof. This is true on the distinguished component S_1 by (4) of the preceding Proposition. Z and W locally invert one another (with respect to composition), so when they are continued to other components of S , they remain inverses. This implies (a). Part (b) follows from (a). \square

Suppose that ϕ and ψ are algebraic homeomorphisms. We would like to understand how to obtain the surface for the composition $\phi \circ \psi$ in terms of the surfaces for ϕ and ψ . The following is at best a heuristic procedure. We are given

$$\begin{array}{ccccc} & \hat{\Sigma}_\phi & & \hat{\Sigma}_\psi & \\ & \swarrow W_\phi & \searrow Z_\phi & \swarrow W_\psi & \searrow Z_\psi \\ D^* \circ D & & D^* \circ D & & D^* \circ D \end{array}$$

Here, in the coordinate Z_ϕ (restricted to the distinguished component S_ϕ of the reflection symmetry), $\phi = W_\phi$, and in the coordinate W_ψ , $\phi^{-1} = Z_\phi$, and similarly for ψ .

Definition 3.

$$\tilde{\Sigma}_{\phi \circ \psi} := \{(q_\phi, q_\psi) \in \hat{\Sigma}_\phi \times \hat{\Sigma}_\psi : Z_\phi(q_\phi) = W_\psi(q_\psi)\}$$

$$\Theta := (\Theta_\phi, \Theta_\psi)$$

$$\tilde{W} := W_\phi \circ pr_\psi \text{ and } \tilde{Z} := Z_\psi \circ pr_\phi$$

The following diagram illustrates the situation:

$$\begin{array}{ccccc} & & \tilde{\Sigma}_\sigma & & \\ & & \swarrow pr_\phi & \searrow pr_\psi & \\ & \hat{\Sigma}_\phi & & \hat{\Sigma}_\psi & \\ & \swarrow W_\phi & \searrow Z_\phi & \swarrow W_\psi & \searrow Z_\psi \\ D^* \circ D & & D^* \circ D & & D^* \circ D \end{array}$$

This surface can be viewed either as the W_ψ pullback of the ramified covering

$$\hat{\Sigma}_\phi \xrightarrow{Z_\phi} D * \circ D$$

or as the Z_ϕ pullback of the ramified covering

$$\hat{\Sigma}_\psi \xrightarrow{W_\psi} D * \circ D$$

Unfortunately it is not clear when $\tilde{\Sigma}_\sigma$ is smooth. So it is not clear that this procedure is of much use.

We next consider some simple examples.

2.2. The Riemann Surface Associated to ϕ_n . Suppose $w_n \neq 0$. The Riemann surface associated to $\phi_n(w_n)$ is essentially defined by the equation

$$w^n(1 + w_n z^n) - (z^n + \bar{w}_n) = 0$$

The affine curve defined by this equation is smooth (the partial derivatives do not simultaneously vanish). However, consider the homogeneous equation

$$Z_0^n Z_2^n + w_n Z_1^n Z_2^n - (Z_0^n Z_1^n + \bar{w}_n Z_0^{2n}) = 0$$

(where $z = Z_1/Z_0$ and $w = Z_2/Z_0$), and the corresponding subvariety in projective space. If $u = Z_0/Z_1$ and $v = Z_2/Z_1$, then

$$u^n v^n + w^n v^n - (u^n + \bar{w}_n u^{2n}) = 0$$

The partial derivatives of the left hand side are

$$\frac{\partial}{\partial u}(LHS) = nu^{n-1}v^n - (nu^{n-1} + \bar{w}_n 2nu^{2n-1})$$

and

$$\frac{\partial}{\partial v}(LHS) = nu^n v^{n-1} + w_n n v^{n-1}$$

Assuming that $n > 1$, these partials vanish simultaneously at $u = v = 0$, and this is a point on the curve. Thus the projective variety defined by the homogeneous equation is not smooth.

Proposition 5. *Suppose that $0 < |w_n| < 1$.*

- (a) *the compact Riemann surface $\hat{\Sigma}$ associated to $\phi_n(w_n)$ has genus $(n-1)^2$.*
- (b) *The antiholomorphic involution Θ for this surface,*

$$\Theta(z, w) = (1/z^*, 1/w^*)$$

has a fixed point set S which consists of n circles; there are $(n-1)(n-2)/2$ holes in the surface on each side of the fixed point set (this is the genus of Σ).

Remarks 2. (a) *This shows the projective variety associated to ϕ_n is not smoothly embedded in \mathbb{P}^2 , for otherwise, using the genus formula for a projective curve (see page 219 of [14]), the genus would be $\frac{1}{2}(2n-1)(2n-2)$, where $2n$ is the degree of the homogeneous polynomial.*

(b) *This should be compared with the Legendre normal form in the theory of Jacobi elliptic functions*

$$y^2 = (1-x^2)(1-k^2x^2)$$

The affine curve is smooth (for $k \neq 0$), but the corresponding projective variety is not smooth, for otherwise the genus would be $\frac{1}{2}(4-1)(4-2) = 3$, and we know the genus is 1.

Proof. Consider first the equation in z, w coordinates:

$$w_n(1 + w_n z^n) - (z^n + \bar{w}_n) = 0$$

The partial derivatives of the left hand side are

$$\frac{\partial}{\partial z}(LHS) = n w_n z^{n-1} w^n - z^{n-1}$$

and

$$\frac{\partial}{\partial w}(LHS) = n w^{n-1} (1 + \bar{w}_n z^n)$$

For points on the affine curve, these are never simultaneously zero, and hence the affine curve is smooth. So we need to know how to compactify this smooth affine algebraic curve. These points are

$$z = \infty, \quad w = \left(\frac{1}{w_n}\right)^{1/n}$$

and these are smooth. To see this, change z to $\frac{1}{\zeta}$. The curve is then

$$w^n = \frac{1 + \bar{w}_n \zeta^n}{\zeta^n + w_n}$$

and this is perfectly well-behaved near $\zeta = 0$. We could alternately have used symmetry to understand the behavior near $z = \infty$, since it is the reflection of what happens at $z = 0$. Consider the holomorphic map

$$z : \hat{\Sigma} \rightarrow \mathbb{C} \cup \{\infty\}$$

Let Σ denote the inverse image of D , the closed unit disk at $z = 0$. We can think of the surface

$$\hat{\Sigma} = \Sigma^* \circ \Sigma$$

as the double of Σ , where the involution Θ is given by (2.4). For the map z , there are $2n$ branch points at the roots $(-\bar{w}_n)^{1/n}$ and their reflections through S^1 . The ramification index is $n - 1$ at each branch point. By the Riemann-Hurwitz relation

$$\chi(\hat{\Sigma}) = n\chi(S^2) - 2n(n - 1) = 2(1 - (n - 1)^2)$$

implying that $\text{genus}(\hat{\Sigma}) = (n - 1)^2$, and the genus of Σ , the number of holes in Σ , is $(n - 1)(n - 2)/2$, since

$$\text{genus}(\hat{\Sigma}) = 2\text{genus}(\Sigma) + n - 1$$

This construction is highly discontinuous at $w_n = 0$. When $w_n = 0$, the curve degenerates to $w^n = z^n$, the Riemann sphere. \square

2.3. $\phi_n \circ \phi_m$. Suppose that $n \neq m$, $w_n, w_m \neq 0$, and $(m, n) = d$. The equation we obtain from $w = \phi_n \circ \phi_m(z)$ is

$$(z^m + \bar{w}_m)^{n/d} (1 - w_n w^n)^{m/d} - (w^n - \bar{w}_n)^{m/d} (1 + w_m z^m)^{n/d} = 0$$

(This arises from setting $Z_{\phi_n} = W_{\phi_m}$ as in Definition 3). On the one hand this polynomial has degree mn/d in each individual variable for all $w_n, w_m \neq 0$. Thus the degree is unchanging. On the one hand the total degree of this polynomial is generically $2mn/d$, but the total degree decreases when $(-w_n)^{m/d} = w_m^{n/d}$. This means that the topology of the surface $\hat{\Sigma}_{\phi_n(w_n) \circ \phi_m(w_m)}$ can vary with the parameters. In particular, for our purposes, it is somewhat of a waste of time to compute the genus. But we will do this anyway.

The values of $z \in \Delta$ at which branching occurs are

$$z^m = -\bar{w}_m \text{ and } \phi_m(z)^n = -\bar{w}_n$$

We want to calculate the ramification for Z at these branch points. For the value $z = (\bar{w}_m)^{1/m}$, there are n inverse images, $(z, w = \bar{w}_n^{1/n})$. By symmetry, the ramification index must be the same at each point, hence this index equals m/d at each of these inverse images. Given z such that $\phi_m(z) = (\bar{w}_n)^{1/n}$, there are m inverse images, and possibly again by symmetry the index is the same at all of them. Hence the ramification index must be n/d at each point. So in a generic situation we expect the ramification index

$$R = 2[m \cdot n \cdot m/d + n \cdot m \cdot n/d]$$

The Riemann-Hurwitz formula now implies

$$\text{genus} = 1 - \frac{mn}{d} + m \cdot n \cdot \frac{m}{d} + n \cdot m \cdot \frac{n}{d}$$

This does not appear to simplify.

2.4. Proof of Theorem 2.2.

Lemma 2. *Suppose that $\sigma = \phi_{i_n}(w_{i_n}) \circ \dots \circ \phi_{i_1}(w_{i_1})$ where $w_i \in \Delta \setminus \{0\}$, $i = i_1, \dots, i_n$, and $i_j \neq i_{j+1}$, $j = 1, \dots, n-1$. Then the degree of Z_σ and W_σ equal*

$$(2.1) \quad \prod_{j=1}^n i_j / \prod_{k=1}^{n-1} (i_k, i_{k+1})$$

In particular given a sequence w with nonvanishing terms, and $\sigma_N = \phi_N(w_N) \circ \dots \circ \phi_1(w_1)$,

$$\text{degree}(Z_{\sigma_N}) = N!$$

Proof. Suppose that $n = 1$, and let $m = i_1$. In this case, in subsection 2.2, we saw that the associated maps $Z, W : \hat{\Sigma}_{\phi_m} \rightarrow \hat{D}$ have degree m . But more simply, in the terminology of chapter 8 of [2], we can view ϕ_m as a branch in a neighborhood of S^1 for the algebraic (multivalued) function

$$(2.2) \quad w = z^{1/m} \circ \phi_m(w_m) \circ z^m$$

(which happens to map $\Delta \rightarrow \Delta$, $S^1 \rightarrow S^1$, and $\Delta^* \rightarrow \Delta^*$). We can calculate the degree by choosing any point $z_0 \in \Delta$ such that $\phi_m(w_m; z_0^m) \neq 0$ (e.g. $z_0 = 0$, because $w_m \neq 0$) and observing that there are exactly m distinct values w_0 such that there exists a (germ of a) branch f of the multivalued expression (2.2) with $f(z_0) = w_0$. Of course we could also consider the “inverse”, and find that given a generic w_0 , there are m corresponding points z_0 . In any event the degree is m .

Similarly the composition $\phi_{i_n}(w_{i_n}) \circ \dots \circ \phi_{i_1}(w_{i_1})$ (where $w_{i_j} \neq 0$ and $i_j \neq i_{j-1}$ for all j) is a branch in a neighborhood of S^1 for the algebraic function

$$w = z^{1/i_n} \circ \phi_1(w_{i_n}) \circ z^{i_n} \circ z^{1/i_{n-1}} \circ \phi_1(w_{i_{n-1}}) \circ \dots \circ z^{1/i_1} \circ \phi_1(w_{i_1}) \circ z^{i_1}$$

or as we prefer,

$$(2.3) \quad w = z^{1/i_n} \circ \phi_1(w_{i_n}) \circ z^{i_n/(i_n, i_{n-1})} \circ z^{1/(i_{n-1}/(i_n, i_{n-1}))} \circ \phi_1(w_{i_{n-1}}) \circ \dots \circ z^{1/(i_1/(i_2, i_1))} \circ \phi_1(w_{i_1}) \circ z^{i_1}$$

To prove the Lemma, it suffices to showing this algebraic function has degree given by the formula (2.1), as we observed in (b) of Proposition 4. We do this by induction on n . We can focus on Δ , because these compositions map Δ into Δ . The degree

is obviously $\leq (2.1)$, so the point is to prove equality. We considered $n = 1$ above. Suppose that $n > 1$. By induction, aside from a finite number of exceptional points in Δ , for $z_0 \in \Delta$ a nonexceptional point, there will be exactly

$$(2.4) \quad \prod_{j=1}^{n-1} i_j / \prod_{k=1}^{n-2} (i_k, i_{k+1})$$

values $w_0 \in \Delta$ such that there is a (germ of a) branch f for

$$(2.5) \quad w_1 = z^{1/i_{n-1}} \circ \phi_1(w_{i_{n-1}}) \circ z^{i_{n-1}/(i_{n-1}, i_{n-2})} \circ \dots \circ z^{1/(i_1/(i_2, i_1))} \circ \phi_1(w_{i_1}) \circ z^{i_1}(z)$$

such that $f(z_0) = w_0$. For given z_0 , the set of w_0 is acted upon by the i_{n-1} roots of unity, and when $w_0 \neq 0$ this action is free. We can perturb z_0 slightly if necessary, so that all of the $w_0 \neq 0$ (we can do this, because the inverse relation has the same properties, so that we can assume the z_0 and w_0 are simultaneously nonexceptional). In this case there will be $1/(i_{n-1}, i_n)$ times (2.4) distinct values w_1 such that there is a (germ of a) branch f for

$$(2.6) \quad w_2 = \phi_{i_n} \circ z^{i_n/i_{n-1}} \circ \phi_1(w_{i_{n-1}}) \circ z^{i_{n-1}/(i_{n-1}, i_{n-2})} \circ \dots \circ z^{1/(i_1/(i_2, i_1))} \circ \phi_1(w_{i_1}) \circ z^{i_1}(z)$$

such that $f(z_0) = w_1$. We can assume that $\phi_{i_n}(w_1^{i_n}) \neq 0$. Then for generic z_0 , there will be (2.1) distinct values w' such that there is a branch f for (2.3) such that $f(z_0) = w'$. Thus the degree for (2.3) is given by (2.1). \square

Remark 4. Note that this formula applies even if for some j , $i_j = i_{j+1}$, provided that $w_{i_j} \neq -w_{i_{j+1}}$.

To prove Theorem 2.2, suppose by way of contradiction that

$$\lambda \phi_{i_n}(w_{i_n}) \circ \dots \circ \phi_{i_1}(w_{i_1})(z) = z, \quad z \in S^1$$

where $\lambda \in S^1$, $w_{i_j} \neq 0$, and $i_j \neq i_{j-1}$ for all j , for some n . This extends to an equality of algebraic functions, and we can consider the degree of both sides. Unless $n = 1$ and $i_1 = 1$, the degree of the left hand side is not equal to 1, the degree of the right hand side. Thus by Lemma 1 (or obviously), $\lambda = 1$ and $w_1 = 0$, a contradiction. This completes the proof of Theorem 2.2.

3. DIFFEOMORPHISMS: PROOF OF THEOREM 0.2

We recall the statement to be proved:

Theorem 3.1. *Fix a permutation $p : \mathbb{N} \rightarrow \mathbb{N} : n \rightarrow n'$. For $s = 1, 2, \dots$ if $w \in \prod_{n=1}^{\infty} \Delta$ and $\sum_{n>0} n^{s-1} |w_n| < \infty$, then the limit*

$$\sigma(p, w; z) = z \prod_{n=1}^{\infty} \frac{(1 + \bar{w}_{n'} \sigma_{n-1}(z)^{-n'})^{1/n'}}{(1 + w_{n'} \sigma_{n-1}(z)^{n'})^{1/n'}}$$

exists and defines a C^s homeomorphism of S^1 .

We first consider the case $s = 1$.

Lemma 3. (a)

$$\Phi'_n(\theta) = \frac{1 - |w_n|^2}{|1 + w_n z^n|^2}, \quad |z| = 1$$

(b)

$$\Sigma'_N(\theta) = \prod_{k=1}^N \Phi'_{k'}(\Sigma_{k-1}(\theta)) = \prod_{k=1}^N \frac{1 - |w_{k'}|^2}{|1 + w_{k'}\sigma_{k-1}^{k'}|^2}$$

(c) If (w_n) is absolutely summable, then the product expression for Σ' ,

$$\Sigma'(\theta) = \prod_{n=1}^{\infty} \frac{1 - |w_{n'}|^2}{|1 + w_{n'}\sigma_{n-1}(z)^{n'}|^2}$$

is absolutely convergent on \mathbb{R} , and σ is a C^1 diffeomorphism of S^1 .*Proof.* (a) is a straightforward calculation. Part (b) follows from the chain rule,

$$\Sigma'_N(\theta) = \prod_{k=1}^N \Phi'_{k'}(\Sigma_{k-1}(\theta))$$

and part (a).

Assuming that (w_n) is absolutely summable, the expression for the derivative of Σ is absolutely convergent, because

$$\prod_{n=1}^{\infty} \frac{1 - |w_{n'}|^2}{|1 + w_{n'}\sigma_{n-1}(z)^{n'}|^2} \leq \prod_{n=1}^{\infty} \frac{1 - |w_n|^2}{(1 - |w_n|)^2} = \prod_{n=1}^{\infty} \frac{1 + |w_n|}{(1 - |w_n|)}$$

The derivative of Σ is positive and continuous; together with the inverse function theorem, this implies that Σ and its inverse are C^1 . \square To investigate the higher derivatives of Σ , define

$$\begin{aligned} B_n(\theta) &:= \ln(\Phi'_n(\theta)) = \ln\left(\frac{1 - |w_n|^2}{|1 + w_n z^n|^2}\right) \\ (3.1) \quad &= -\ln(1 + \bar{w}_n z^{-n}) + \ln(1 - |w_n|^2) + \ln(1 + w_n z^n), \quad z = e^{i\theta} \end{aligned}$$

and

$$B(\theta) := \ln(\Sigma'(\theta)) = \sum_{n=1}^{\infty} B_{n'}(\Sigma_{n-1}(\theta))$$

Lemma 4. (a) For $s = 1, 2, \dots$,

$$B_n^{(s)}(\theta) = (in)^s \frac{w_n z^n A_{s-1}(-w_n z^n)}{(1 + w_n z^n)^s} + c.c., \quad z = e^{i\theta}$$

where the A_{s-1} are the Eulerian polynomials.(b) For given s there is a constant $c = c(s)$ independent of n such that

$$|B_n^{(s)}(\Sigma_{n-1}(\theta))| \leq cn^s |w_n| (1 - |w_n|)^{-s}$$

Proof. From (3.1) (and expanding the logarithm in a power series)

$$\begin{aligned} \left(\frac{\partial}{\partial \theta}\right)^s B_n(\theta) &= \left(\frac{\partial}{\partial \theta}\right)^s \ln(1 + w_n z^n) + c.c. \\ &= \sum_{k=1}^{\infty} \frac{1}{k} (-w_n)^k \left(\frac{\partial}{\partial \theta}\right)^s z^{kn} + c.c. = (in)^s \sum_{k=1}^{\infty} k^{s-1} (-w_n z^n)^k + c.c. \end{aligned}$$

This can be summed using the basic power series identity of Euler

$$(3.2) \quad \sum_{k=1}^{\infty} k^n q^k = \frac{q A_n(q)}{(1 - q)^{n+1}}, \quad |q| < 1$$

where A_n is the n th Eulerian polynomial. This implies part (a).

Part (b) follows from (a), where we bound $|z^n A_{s-1}(w_n z^n)|$ by a constant depending only on s (and the size of coefficients for the Eulerian polynomial A_{s-1}), using the facts that $|z| = 1$ and $|w_n z^n| < 1$. \square

We now complete the proof of Theorem 3.1.

Proof. We will prove the slightly broader statement that if $\sum n^{s-1}|w_n| < \infty$, then there is a bound for the derivatives of B_N up to order $s-1$ which is independent of N . This will imply that B itself is C^{s-1} . Lemma 3 takes care of the case $s = 1$.

Suppose $s > 1$. Faa di Bruno's formula for higher derivatives of a composition of functions implies that

$$(3.3) \quad \begin{aligned} \left(\frac{d}{d\theta}\right)^{s-1} B_N(\theta) &= \sum_{n=1}^N \left(\frac{d}{d\theta}\right)^{s-1} (B_{n'} \circ \Sigma_{n-1})(\theta) \\ &= \sum_{n=1}^N \sum_{k=1}^{s-1} B_{n'}^{(k)}(\Sigma_{n-1}(\theta)) \mathcal{B}_{s-1,k}(\Sigma'_{n-1}, \dots, \Sigma_{n-1}^{(s-1-k)}) \end{aligned}$$

where $\mathcal{B}_{s-1,k}$ denotes the Bell polynomial of degree k . For example

$$B''(\theta) = \sum_{n=1}^{\infty} (B_{n'}''(\Sigma_{n-1}(\theta)) \Sigma'_{n-1}(\theta)^2 + B_{n'}'(\Sigma_{n-1}(\theta)) \Sigma_{n-1}''(\theta))$$

In general the Bell polynomials have positive integral coefficients.

Using (b) of Lemma 4, we can bound the sum in (3.3) by

$$\begin{aligned} &\sum_{n=1}^{\infty} \sum_{k=1}^{s-1} c n'^k \frac{|w_{n'}|}{(1 - |w_{n'}|)^k} B_{s-1,k}(\sup |\Sigma'_{n-1}|, \dots, \sup |\Sigma_{n-1}^{(s-1-k)}|) \\ &\leq \sum_{n=1}^{\infty} c n'^{s-1} |w_{n'}| \sum_{k=1}^{s-1} B_{s-1,k}(\sup |\Sigma'_{n-1}|, \dots, \sup |\Sigma_{n-1}^{(s-1-k)}|) \end{aligned}$$

In this sum, because s is fixed, we are considering a fixed finite number of Bell polynomials. Since the orders of the derivatives appearing in the sum over k are strictly less than $s-1$, by induction we find a bound for

$$\sum_{k=1}^{s-1} B_{s-1,k}(\sup |\Sigma'_{n-1}|, \dots, \sup |\Sigma_{n-1}^{(s-1-k)}|)$$

which is independent of N . This completes the induction step. \square

4. DIFFEOMORPHISMS: DISCUSSION OF QUESTION 2

Throughout this section we fix a permutation of positive integers, $p : n \rightarrow n'$.

Question. Does the map

$$S^1 \times \left(c^\infty \cap \prod_{n=1}^{\infty} \Delta \right) \rightarrow \text{Diff}(S^1) : (\lambda; w) \rightarrow \sigma(p, w; z)$$

define a bijection, where c^∞ is the Frechet space of rapidly decreasing sequences?

In the theory of Fourier series, $f = \sum f_n \sin(n\theta)$, the n th coefficient can be obtained directly by solving a minimization problem, involving Euclidean distance. To find the inverse to the map in the question, we will try to solve an ordered sequence of minimization problems (depending on p). In place of Euclidean length, we will use a height function. There is some freedom in the choice of this function. In the first part of this section, for technical ease, we will use a vacuum expectation, depending on the choice of a representation. However, there appears to be a preferred choice, namely

$$(4.1) \quad H : QS(S^1) \rightarrow \mathbb{R}_{\geq 0} : \phi \rightarrow -\log(a(\phi))$$

where $\phi = lmau$ is the unique triangular factorization. This function (or an extension which we consider in the third subsection) possibly has the natural maximal domain for our problem. We will pursue this in a tentative way in the second and third subsections below (the reader is forewarned that we have made disappointing progress).

4.1. Vacuum Expectations. We will discuss several conjectural lemmas. For the proofs of these lemmas we will consider an irreducible unitary positive energy representation for the Virasoro group $\widehat{Diff}(S^1)$ with parameters (c, h) . At the Lie algebra level this representation is an irreducible lowest weight representation of the Virasoro algebra which is unitarizable; see [13], [16], [22] and [31] for issues regarding globalization of such representations. To avoid issues involving singular vectors, we will assume that $c > 1$ and $h > 0$, unless stated otherwise (the preferred height function (4.1) corresponds to $c = 0, h = 1$, and the corresponding representation is not unitary). The height function we consider is the vacuum expectation

$$H_{c,h} : Diff(S^1) \rightarrow \mathbb{R}_{\geq 0} : \phi \rightarrow -\log|\langle \tilde{\phi}v, v \rangle|^2$$

Here v denotes the normalized vacuum (or unit lowest energy state) for the representation, the dependence of the inner product on the parameters is implicit, and $\tilde{\phi} \in \widehat{Diff}(S^1)$ maps to $\phi \in Diff(S^1)$. For example if $(c, h) = (1, 0)$, then

$$H_{1,0}(\phi) = -\log(\det(A_a(\phi)^* A_a(\phi)))$$

which is defined on the critical group of $W^{1+1/2}$ homeomorphisms, and which is smaller than $QS(S^1)$. But for purposes of this subsection, we only need to consider diffeomorphisms - subtle domain issues are not relevant.

Remark 5. Suppose that H is a Hilbert space. The Riemannian distance for the Fubini-Study metric for the projective space $\mathbb{P}(H)$ is given by

$$dist_{\mathbb{P}}(\mathbb{P}(\xi), \mathbb{P}(\eta)) = \arccos|\langle \xi, \eta \rangle|$$

where $|\xi| = |\eta| = 1$. In the following we could just as well use $dist_{\mathbb{P}}(\tilde{\phi}v, v)$ as the height function.

Question 5. Suppose that $n > 0$.

(a) Given $\phi \in Diff(S^1)$, does the function

$$\Delta \rightarrow \mathbb{R}_{\geq 0} : w \rightarrow H(\phi \circ \phi_n(-w))$$

achieve its infimum at a unique point $w_n \in \Delta$?

(b) Assuming the truth of (a), is the function $w_n : Diff(S^1) \rightarrow \Delta$ continuous?

Proof. When $|w| \rightarrow 1$, $\phi_n(-w)$ is tending to infinity in the group $PSU(1,1)^{(n)}$. Consequently a matrix coefficient such as

$$\langle \tilde{\phi} \circ \phi_n(-w)v, v \rangle = \langle \phi_n(-w)v, \tilde{\phi}^{-1}v \rangle$$

tends to zero as $|w| \rightarrow 1$ (for a relatively direct proof, see Section 2.4 of [35]). Thus $H(w)$ goes to infinity when w approaches the boundary S^1 . Since $H \geq 0$, this implies that H does achieve its infimum in Δ . The question is whether there is a unique point at which the minimum is attained.

It is convenient to slightly reformulate the problem. The set $\{\lambda \phi_n(-w)v : \lambda \in S^1, w \in \Delta\}$ is the same as $PSU(1,1)^{(n)} \cdot v$. We are interested in the function

$$PSU(1,1)^{(n)} \rightarrow \mathbb{R} : g \rightarrow |\langle g \cdot v, \tilde{\phi}^{-1}v \rangle|^2$$

where we are viewing $PSU(1,1)^{(n)}$ as a subgroup of $\widetilde{Diff}(S^1)$. This function is really defined on the disk $PSU(1,1)^{(n)}/Rot$, but it is more convenient to work with the group. By considering a first variation $g(t) = ge^{tL}$, we see a critical point g satisfies

$$Re(\langle g \cdot L \cdot v, \tilde{\phi}^{-1}v \rangle \langle g \cdot v, \tilde{\phi}^{-1}v \rangle^*) = 0$$

as L ranges over the Lie algebra of $PSU(1,1)^{(n)}$, which we calculated in (1.2). This equation is automatically satisfied by $L = iL_0$, because the function is really defined on the quotient disk. Because $L_{-n} \cdot v = 0$, the two real equations for $L = L_n - L_{-n}$ and $L = i(L_n + L_{-n})$ are equivalent to a single complex equation for a critical point,

$$\langle g \cdot L_n \cdot v, \tilde{\phi}^{-1}v \rangle = 0$$

Now suppose that we consider a second variation $g(s, t) = ge^{sL'}e^{tL}$. At a critical point the Hessian is

$$(L', L) \rightarrow \langle g \cdot L' L \cdot v, \tilde{\phi}^{-1} \cdot v \rangle \langle g \cdot v, \tilde{\phi}^{-1} \cdot v \rangle^* + c.c.$$

We would like to show that this is necessarily negative definite. In particular we must show the following is negative:

$$\begin{aligned} & \langle g \cdot (L_n - L_{-n})^2 \cdot v, \tilde{\phi}^{-1} \cdot v \rangle \langle g \cdot v, \tilde{\phi}^{-1} \cdot v \rangle^* + c.c. \\ &= \langle g \cdot L_n^2 \cdot v, \tilde{\phi}^{-1} \cdot v \rangle \langle g \cdot v, \tilde{\phi}^{-1} \cdot v \rangle^* + c.c. - (2nh + \frac{c}{12}n(n^2 - 1))|\langle g \cdot v, \tilde{\phi}^{-1} \cdot v \rangle|^2 \end{aligned}$$

In a similar way

$$\begin{aligned} & \langle g \cdot (i(L_n + L_{-n}))^2 \cdot v, \tilde{\phi}^{-1} \cdot v \rangle \langle g \cdot v, \tilde{\phi}^{-1} \cdot v \rangle^* + c.c. \\ &= -\langle g \cdot L_n^2 \cdot v, \tilde{\phi}^{-1} \cdot v \rangle \langle g \cdot v, \tilde{\phi}^{-1} \cdot v \rangle^* + c.c. - (2nh + \frac{c}{12}n(n^2 - 1))|\langle g \cdot v, \tilde{\phi}^{-1} \cdot v \rangle|^2 \end{aligned}$$

$$\begin{aligned} & \langle g \cdot i(L_n + L_{-n})(L_n - L_{-n}) \cdot v, \tilde{\phi}^{-1} \cdot v \rangle \langle g \cdot v, \tilde{\phi}^{-1} \cdot v \rangle^* + c.c. \\ &= i\langle g \cdot L_n^2 \cdot v, \tilde{\phi}^{-1} \cdot v \rangle \langle g \cdot v, \tilde{\phi}^{-1} \cdot v \rangle^* + c.c. \end{aligned}$$

Relative to the basis $e - f, i(e + f)$, the matrix of the Hessian, decomposed into a diagonal and traceless part, is

$$\begin{aligned} & -(\frac{2}{n}h + \frac{c}{12n}(n^2 - 1))|\langle g \cdot v, \tilde{\phi}^{-1} \cdot v \rangle|^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ & + (\langle g \cdot \frac{1}{n^2}L_n^2 \cdot v, \tilde{\phi}^{-1} \cdot v \rangle \langle g \cdot v, \tilde{\phi}^{-1} \cdot v \rangle^* + c.c.) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

It is straightforward to compute that

$$|L_n^2 v|^2 = 2(2nh + \frac{c}{12}n(n^2 - 1))^2 + 2n^2(2nh + \frac{c}{12}n(n^2 - 1))$$

But this is not good enough to show that the traceless part is dominated by the diagonal part.

If we consider the matrix for the Hessian for the original height function $H_{c,h}$, this is given by

$$\begin{aligned} & -(\frac{2}{n}h + \frac{c}{12n}(n^2 - 1)) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ & + \frac{1}{n^2} \frac{\langle g \cdot L_n^2 \cdot v, \tilde{\phi}^{-1} \cdot v \rangle}{\langle g \cdot v, \tilde{\phi}^{-1} \cdot v \rangle} + c.c.) \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \end{aligned}$$

This is a very interesting expression. In particular we see (what is easy to see from an abstract point of view)

$$\partial \bar{\partial} H_{c,h} = (\frac{2}{n}h + \frac{c}{12n}(n^2 - 1)) \frac{dw \wedge d\bar{w}}{(1 - w\bar{w})^2}$$

So $H_{c,h}$ is subharmonic. In particular there are not any local maxima. We still have to rule out saddle points. We have not succeeded at doing this. \square

Now suppose that $\phi \in \text{Diff}(S^1)$. Assuming an affirmative answer to the proceeding question, by applying this $n = 1'$, then we obtain $w_{1'}$ which minimizes $H(\phi \circ \phi_{1'}(-w))$. If we apply this again with $\phi \circ \phi_{1'}(-w_{1'})$ in place of ϕ , then we obtain $w_{2'}$ which minimizes $H(\phi \circ \phi_{1'}(-w) \circ \phi_{2'}(-w))$, and so on. Hence given ϕ , we obtain a $w \in \prod_{n=1}^{\infty} \Delta$ such that $H(\phi \circ \sigma_N(p, w)^{-1})$ is a nonincreasing sequence of nonnegative numbers.

It is important to keep in mind that the sequence we obtain a priori depends on the choice of height function, hence (c, h) . At any rate we would be able to assert that the induced map

$$\text{Diff}(S^1) \rightarrow \prod_{n=1}^{\infty} \Delta : \phi \rightarrow w$$

is continuous for the product topology on the target, i.e. each w_n is a continuous function of ϕ .

Question. Suppose that $\phi = \sigma_N(p, \tilde{w})$ for some $\tilde{w} \in \prod_{n=1}^{\infty} \Delta$. Is $\tilde{w} = w$, where w is the sequence we obtain in the preceding paragraphs?

The answer to this appears to be negative in general. We would have to show that the function

$$|\langle \sigma_N(p, \tilde{w}) \circ \phi_{1'}(-w)v, v \rangle|$$

achieves it maximum when $w = w_{1'}$. This function equals

$$\begin{aligned} & |\langle \phi_{1'}(w_{1'}) \circ \phi_{1'}(-w)v, \phi_{2'}(-w_{2'}) \circ \dots \circ \phi_{N'}(-w_{N'})v \rangle| \\ & = |\langle \phi_{1'}(u)v, \phi_{2'}(-w_{2'}) \circ \dots \circ \phi_{N'}(-w_{N'})v \rangle| \end{aligned}$$

where $u = \phi_{1'}(-\tilde{w}; w_{1'})$. We have to show that this is maximized when $u = 0$. This appears to not be true. To see this, suppose that

$$\langle \phi_3 \phi_2 L_1 v, v \rangle = 0$$

for any parameters w_3, w_2 . By differentiating, we obtain

$$\langle (L_3 - L_{-3})(L_2 - L_{-2})L_1 v, v \rangle = 0$$

The left hand side equals

$$\begin{aligned} -\langle L_{-3}L_2L_1v, v \rangle &= -\langle (L_2L_{-3} + [L_{-3}, L_2])L_1v, v \rangle \\ &= -5\langle L_{-1}L_1v, v \rangle = -10\langle L_0v, v \rangle \end{aligned}$$

and this is basically never zero.

4.2. A More Natural Approach. In this subsection we consider

$$H : QS(S^1) \rightarrow \mathbb{R}_{\geq 0} : \phi \rightarrow -\log(a(\phi))$$

where $\phi = lmau$ is the unique triangular factorization.

There is a useful formula (essentially the well-known area theorem, see part (e) of Proposition 1 of [6])

$$a(\phi) = \left(\frac{1 - \sum_{m=1}^{\infty} (m-1)|b_m|^2}{1 + \sum_{n=1}^{\infty} (n+1)|u_n|^2} \right)^{1/2} \leq 1$$

This for example shows that $H \geq 0$, and $H(\phi) = 0$ if and only if ϕ is a rotation.

Question 6. Suppose that $n > 0$.

(a) Given $\phi \in QS(S^1)$, does the function

$$\Delta \rightarrow \mathbb{R}_{\geq 0} : w \rightarrow H(\phi \circ \phi_n(-w))$$

achieves its infimum at a unique point $w_n \in \Delta$?

(b) Is the function $w_n : QS(S^1) \rightarrow \Delta$ is continuous?

It is instructive to first consider the case $n = 1$. In this case (by Proposition 1) there is an explicit formula

$$H(w) = -\log(a(\phi)(1 - w\bar{w})|u'(-\bar{w})|)$$

where $\phi = lmau$ is the unique triangular factorization for ϕ .

As in the previous subsection, the first step is to show that $H(w)$ goes to infinity when w approaches the boundary S^1 . To prove this we must show that $(1 - w\bar{w})|u'(-\bar{w})|$ goes to 0 as w approaches the boundary. We first note that this quantity is bounded. By the Cauchy integral formula (and noting the appearance of the Poisson kernel)

$$\begin{aligned} (1 - w\bar{w})|u'(-\bar{w})| &= \frac{1}{2\pi} \left| \int_{S^1} \frac{(1 - |w|^2)u(\zeta)}{(\bar{w} - \zeta)^2} d\zeta \right| \\ &\leq |u|_{L^\infty(S^1)} \frac{1}{2\pi} \int_{S^1} \frac{(1 - |w|^2)}{(1 - 2|w|\cos(\theta) - |w|^2)} d\theta = |u|_{L^\infty} \end{aligned}$$

(Note that we are only using the boundedness of u and not quasisymmetry). So we at least have boundedness.

In the present context, because u is quasiconformal, there should be some refinement which shows that we actually have decay at infinity.

Assuming decay at infinity, since $H \geq 0$, this implies that H does achieve its infimum in Δ . The question then is whether there is a unique point at which the minimum is attained.

The derivative of H (which is real valued) is given by

$$dH = \left(\frac{w}{1 - w\bar{w}} + \frac{1}{2} \frac{u''(-\bar{w})}{u'(-\bar{w})} \right) d\bar{w} + c.c.$$

and

$$\partial\bar{\partial}H = \frac{dw \wedge d\bar{w}}{(1 - w\bar{w})^2}$$

At a critical point $w \in \Delta$ for H ,

$$\frac{u''(-\bar{w})}{u'(-\bar{w})} = -\frac{2w}{1 - w\bar{w}}$$

Remark 6. (a) Suppose that $\phi = \phi_n(w_n)$. In this case $u(z) = \frac{z}{(1 + w_n z^n)^{-1/n}}$ and

$$\frac{u''(-\bar{w})}{u'(-\bar{w})} = -\frac{(n+1)w_n(-\bar{w})^{n-1}}{1 + w_n(-\bar{w})^n}$$

One can then verify that when $n = 1$, $w = w_1$, and when $n > 1$, $w = 0$, are the unique critical points.

(b) Suppose that $u \in \mathcal{S}$, the set of normalized univalent functions in Δ . We can consider the analogue of the H function

$$H(w) = -\log(1 - w\bar{w}) - \log|u(-\bar{w})|, \quad w \in \Delta$$

The derivative of H and the Laplacian of H are given by the same formulas as above. We can ask the following questions:

(1) Is H bounded below? In general the answer is no. If u is the Koebe function, then $u'(z) = \frac{1+z}{(1-z)^3}$, and H goes to $-\infty$ as $w \rightarrow 1$. If $u = \frac{z}{1-z}$, then

$$H = \log \frac{|1 + \bar{w}|^2}{1 - w\bar{w}}$$

which goes to $-\infty$ as $w \rightarrow -1$.

(2) Does H have a critical point in the disk? This can be reformulated in the following way (see [15], page 351). Given u and $w_0 \in \Delta$, there exist unique constants C_i such that

$$f = C_1 + C_2 u \circ \phi_1(\bar{w}_0) \in \mathcal{S}$$

and the first nontrivial coefficient is

$$f_1 = w_0 + \frac{1}{2}(1 - w_0\bar{w}_0) \frac{u''(-\bar{w}_0)}{u'(-\bar{w}_0)}$$

Thus we are asking, can we always find a w_0 such that this first coefficient vanishes, and is there a unique such w_0 ?

For the Koebe function the answer is yes; the unique critical point is $w_0 = -1/3$. But for $u = \frac{z}{1-z}$, there does not exist a critical point in the disk.

(3) If there is a critical point in the disk, is it unique? This is unclear.

In any event we do need to identify the minimal assumption on u that will guarantee that H is bounded below and that will force H to go to $+\infty$ as w tends to the boundary. Merely assuming u is bounded does not seem enough. u quasymmetric should work. Is it enough for u to simply arise from some welding? This seems unlikely.

Now suppose $n > 1$. As w goes to infinity, it seems very plausible that $H \uparrow \infty$. So again H should achieve its infimum in Δ . It remains to show that there is a unique global minimum. The main formula we have to work with is

$$\partial\bar{\partial}H = \frac{1}{n} \frac{dw \wedge d\bar{w}}{(1 - w\bar{w})^2}$$

This implies that H is subharmonic, and in particular H does not have a local maximum. Since H goes to infinity at the boundary, if there is more than one minimum, then there must be a saddle point. But it is not clear how to rule out saddle points.

4.3. A Possible Further Extension. Not all homeomorphisms have triangular factorizations, and in some sense these are typically not unique (see[5]). Consequently it is natural to think of the parameters w_n as functions of loops in the plane (i.e. triangular factorizations), rather than homeomorphisms. This leads to technical issues which are way beyond my ability to resolve, but I want to at least put the problem in the right context.

Suppose that $\gamma \in \text{Loop}^1(\mathbb{C} \setminus \{0\})$, the set of self-avoiding loops which surround zero. By the Jordan curve theorem the complement of γ in $\mathbb{C} \cup \{\infty\}$ has two connected components, U_{\pm} , so that

$$\mathbb{C} \cup \{\infty\} = U_+ \sqcup \gamma \sqcup U_-$$

where $0 \in U_+$ and $\infty \in U_-$. There are based conformal isomorphisms

$$\phi_+ : (\Delta, 0) \rightarrow (U_+, 0), \quad \phi_- : (\Delta^*, \infty) \rightarrow (U_-, \infty)$$

The map ϕ_- can be uniquely determined by normalizing the Laurent expansion in $|z| > 1$ to be of the form

$$(4.2) \quad \phi_-(z) = \rho_{\infty}(\gamma)L(z), \quad L(z) = z(1 + \sum_{n \geq 1} b_n z^{-n})$$

where $\rho_{\infty}(\gamma) > 0$ is the transfinite diameter (see chapters 16 and 17 of [15] for numerous formulas for ρ_{∞}). The map ϕ_+ can be similarly uniquely determined by normalizing its Taylor expansion to be of the form

$$(4.3) \quad \phi_+(z) = \rho_0(\gamma)u(z), \quad u(z) = z(1 + \sum_{n \geq 1} u_n z^n)$$

where $\rho_0(\gamma) > 0$ is called the conformal radius with respect to 0. By a theorem of Carathéodory (see Theorem 17.5.3 of [15]), both ϕ_{\pm} extend uniquely to homeomorphisms of the closures of their domain and target. This implies that the restrictions $\phi_{\pm} : S^1 \rightarrow \gamma$ are topological isomorphisms. Thus there is a well-defined welding map

$$(4.4) \quad W : \text{Loop}^1(\mathbb{C} \setminus \{0\}) \rightarrow \{\sigma \in \text{Homeo}^+(S^1) : \sigma = lau\} \times \mathbb{R}^+ : \gamma \mapsto (\sigma(\gamma), \rho_{\infty}(\gamma))$$

where

$$\sigma(\gamma, z) := \phi_-^{-1}(\phi_+(z)) = lau, \quad a(\gamma) = \frac{\rho_0(\gamma)}{\rho_{\infty}(\gamma)}$$

and l is the inverse mapping for L .

The first thing to observe is that in this context the height function extends and is given by the formula (remarkably similar to ())

$$H : \text{Loop}^1(\mathbb{C} \setminus \{0\}) \rightarrow \mathbb{R}_{\geq 0} : \gamma \mapsto -\log(a) = -\log(\rho_0) + \log(\rho_{\infty})$$

We know how to calculate variations of this kind of thing.

Question 7. *Is there an extension*

$$Loop^1(\mathbb{C} \setminus \{0\}) \rightarrow \prod_{n=1}^{\infty} \Delta : \gamma \rightarrow w$$

which is characterized by the fact that $H(\sigma(\gamma) \circ \sigma_{N-1}(w_1, \dots, w_{N-1})^{-1} \circ \phi_N(-w))$ achieves its minimum at $w = w_N$ for $N = 1, 2, \dots$?

It strikes me as likely that the answer to this naive question is negative, and more likely this extension exists only in an almost sure sense relative to Werner's measure (and its generalizations). The follow up question is whether these coordinates, assuming they exist, are independent random variables.

5. LESS REGULAR HOMEOMORPHISMS

Throughout this section we fix a permutation $p : n \rightarrow n'$ of the natural numbers. We also suppose that $\sum_{n>0} \frac{1}{n} |w_n| < \infty$. In this situation, as we noted in the Introduction, $\sigma(p, w) : S^1 \rightarrow S^1$ is a continuous degree one map. The problem is to determine when this function is invertible, or at least find a reasonably sharp sufficient condition. The key is to understand the derivative.

As in the introduction, we write $\sigma(p, w; e^{i\theta}) = \exp(i\Sigma(\theta))$, where $\Sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous nondecreasing function satisfying $\Sigma(\theta + 2\pi) = \Sigma(\theta) + 2\pi$. Using (0.3) we can take

$$\Sigma(\theta) = \theta + \sum_{n=1}^{\infty} \frac{1}{n'} \Theta(1 + w_{n'} \sigma_{n-1}(z)^{n'})$$

But we will not make much use of this expression.

Let $d\Sigma$ denote the derivative of Σ in the sense of generalized functions; we can interpret $\frac{1}{2\pi} d\Sigma$ as a probability measure. Since Σ is nondecreasing, there is a unique orthogonal decomposition

$$\frac{d\Sigma}{2\pi} = \Sigma'(\theta) \frac{d\theta}{2\pi} + \mu_s$$

where the pointwise derivative Σ' exists a.e. and μ_s is singular with respect to $d\theta$.

Lemma 5. (a) *Suppose that $w \in l^2 \cap \prod_{n=1}^{\infty} \Delta$. Then $\{\sigma^n : n = 0, 1, 2, \dots\}$ is an orthonormal set with respect to the probability measure $\frac{1}{2\pi} d\Sigma$.*

Conjecture 3. (b) *The sum*

$$\sum_{n=1}^{\infty} w_{n'} \sigma_{n-1}(z)^{n'}$$

converges in $L^2(d\Sigma)$.

(c) *The sum*

$$\sum_{n=1}^{\infty} w_{n'} \log(1 + w_{n'} \sigma_{n-1}(z)^{n'})$$

converges in $L^2(d\Sigma)$.

Proof. (a) This is obviously true whenever σ is invertible, because by changing z to $\sigma^{-1}(z)$

$$\int_{S^1} \sigma(z)^n (\sigma(z)^m)^* d\Sigma(\theta) = \int_{S^1} z^n (z^m)^* d\theta = 2\pi\delta(n-m)$$

We need to justify the limiting claim. We need to be able to take a limit

$$\int_{S^1} \sigma(w_\epsilon)^n (\sigma(w_\epsilon)^m)^* d\Sigma(w_\epsilon)$$

where $w_\epsilon \rightarrow w$ in l^2 as $\epsilon \rightarrow 0$. The integrand is converging uniformly to $\sigma(z)^n (\sigma(z)^m)^*$. So we need to know that the measures are converging to $d\Sigma(w)$ in the weak star topology for probability measures on S^1 . This is equivalent to pointwise convergence of the corresponding cumulative distributions functions at all points of continuity (see Theorem 2.1 of [4]). This is satisfied in our context, because $w_\epsilon \rightarrow w$ in l^2 implies that $\Sigma(w_\epsilon) \rightarrow \Sigma(w)$ uniformly.

(b) Part (a) implies that $\sum_{n \geq 0} w_{n'} \sigma^{n'} \in L^2(d\Sigma)$.

$$\sum_{n=1}^{\infty} w_{n'} \sigma_{n-1}(z)^{n'} = \sum_{n=1}^{\infty} w_{n'} \left(\frac{\sigma_{n-1}(z)}{\sigma(z)} \right)^{n'} \sigma(z)^{n'}$$

What we want to argue is that

$$\left(\frac{\sigma_{n-1}(z)}{\sigma(z)} \right)^{n'} = \exp(-2i \sum_{k=n}^{\infty} \frac{n'}{k'} \Theta(1 + w'_k \sigma_{k-1}^{k'}))$$

is basically just a constant of absolute value one, for large n . This is a subtle issue which I have not succeeded at understanding in a useful way.

(c) Since

$$\log(1 + w_{n'} \sigma_{n-1}(z)^{n'}) = w_{n'} \sigma_{n-1}(z)^{n'} + O(|w_{n'}|^2)$$

and $|w_n|^2$ is absolutely summable, this part would follow from (b). \square

Question 8. If $w \in l^2 \cap \prod_{n=1}^{\infty} \Delta$, then

$$\frac{d\theta}{2\pi} = \left(\prod_{n=1}^{\infty} (1 - |w_{n'}|^2) \right)^{-1} \exp(2 \sum_{n=1}^{\infty} \log(|1 + w_{n'} \sigma_{n-1}(z)^{n'}|) \frac{d\Sigma}{2\pi})$$

Let

$$\exp(f_N) := \prod_{n=1}^N \frac{1 - |w_{n'}|^2}{|1 + w_{n'} \sigma_{n-1}(z)^{n'}|^2}$$

and

$$\exp(f) := \left(\prod_{n=1}^{\infty} (1 - |w_{n'}|^2) \right) \exp(-2 \sum_{n=1}^{\infty} \log(|1 + w_{n'} \sigma_{n-1}(z)^{n'}|))$$

Then $f_N \rightarrow f$ in $L^2(d\Sigma)$, $d\Sigma_N \rightarrow d\Sigma$ in the weak star topology, and for all N

$$\frac{d\theta}{2\pi} = \exp(-f_N) \frac{d\Sigma_N}{2\pi}$$

We have a family of probability measures

$$\frac{1}{\mathfrak{z}_{N,M}} \exp(-f_N) \frac{d\Sigma_M}{2\pi}$$

If we fix N and let $M \rightarrow \infty$, then we get

$$\frac{1}{\mathfrak{z}_{N,\infty}} \exp(-f_N) \frac{d\Sigma}{2\pi}$$

If we fix M and let $N \rightarrow \infty$, then we possibly get nonsense

$$\frac{1}{\mathfrak{z}_{\infty,M}} \exp(-f) \frac{d\Sigma_M}{2\pi}$$

So we require $M \geq N$. The idea is that this limit should imply that $\exp(-f) \frac{d\Sigma}{2\pi}$ is rotation invariant. Thus there is a constant c such that

$$\exp(-f) \frac{d\Sigma}{2\pi} = c \frac{d\theta}{2\pi}$$

Fatou's Lemma implies $c = \int \exp(-f) \frac{d\Sigma}{2\pi} \leq 1$ Because f is L^2 , $0 < c < 1$. We then have to show that $c = 1$, and this is not clear.

Question 9. Fix the permutation p as above. If $w \in l^2$, then $\sigma(p, w) := \lim_{N \rightarrow \infty} \sigma_N \in \text{Homeo}(S^1)$.

Assuming the answer to Question 8 is yes, in the notation above,

$$\exp(-f) \left(\Sigma' \frac{d\theta}{2\pi} + \mu_s \right) = \frac{d\theta}{2\pi}$$

This implies that $\exp(-f) \Sigma' = 1$ a.e. $[d\theta]$. Thus $\Sigma' = \exp(f) > 0$ a.e. $[d\theta]$. And this would imply that Σ is increasing and hence is a homeomorphism.

5.1. Examples of Noninvertibility. We need a concrete counterexample to the claim that summability of $\frac{1}{n}|w_n|$ implies that σ is invertible. Here are vague thoughts about how to obtain an example. The derivative of Σ_N is given by

$$\Sigma'_N(\theta) = \prod_{n=1}^N \frac{1 - |w_n|^2}{|1 + w_n \sigma_{n-1}(z)^n|^2}$$

where $z = \exp(i\theta)$. Suppose that all of the $w_n > 0$. Then

$$\Sigma'_N(0) = \prod_{n=1}^N \frac{1 - |w_n|^2}{|1 + w_n|^2} \leq \prod_{n=1}^N (1 - |w_n|^2)$$

and this clearly goes to zero when w is not square summable. When one numerically considers the graph of Σ_N what one sees is that the graph is in fact flat in an entire neighborhood of $\theta = 0$. Thus it appears that when w is not square summable, it is very difficult for σ to be invertible. The reason for this is the following. Fix θ and consider $1 + w_n \sigma_{n-1}(z)^n$ for large n , where we think of w_n as reasonably small. Because of the large power, the values of $\sigma_{n-1}(z)^n$ are plausibly randomly distributed around the circle. This means that about half of the time $1 + w_n \sigma_{n-1}(z)^n$ will have magnitude bigger than one. Thus by the same estimate it appears that the derivative will go to zero. Thus σ will not be invertible. Note that if we take $\theta = \pi$ (again assuming the $w_n > 0$)

$$\Sigma'_N(\pi) = \prod_{n=1}^N \frac{1 - |w_n|^2}{|1 - w_n|^2} = \prod_{n=1}^N \frac{1 + |w_n|}{(1 - |w_n|)}$$

This is diverging. Thus the mass of the probability measure is concentrating in the vicinity of $\theta = \pi$, as one sees numerically.

6. SEMIGROUP OF INCREASING FUNCTIONS

In Subsection 5.1 we have seen that when (w_n) is not square summable, it may happen that $\sigma(w)$ is not invertible. For this reason it makes sense to consider the increasing function $\Sigma(w)$ (modulo $w\pi\mathbb{Z}$) and think about algebraic structures that remain in the absence of invertibility.

Definition 4. (a) $\widetilde{CDF}(S^1)$ is the semigroup of right continuous nondecreasing functions on \mathbb{R} satisfying

$$\Sigma(\theta + 2\pi) = \Sigma(\theta) + 2\pi$$

where multiplication is given by composition.

(b) $CDF(S^1)$ is the quotient of $\widetilde{CDF}(S^1)$ by the central subgroup $2\pi\mathbb{Z}$, where $2\pi n$ is identified with the map $\theta \rightarrow \theta + 2\pi n$.

Proposition 6. (a) The map

$$CDF(S^1) \rightarrow Prob(S^1) : \Sigma \rightarrow \frac{1}{2\pi}d\Sigma$$

the distributional derivative, is an isomorphism of sets.

(b) With the weak star topology relative to $C^0(S^1)$, $CDF(S^1)$ is a topological semigroup.

(c) $Homeo(S^1)$ is the group of units for $CDF(S^1)$. It is not dense. It is not closed.

(d) The cdfs corresponding to measures with finite support is a dense normal subsemigroup.

(e) Fix n . The cdfs corresponding to measures with n atoms is a normal subsemigroup.

Proof. This is relatively straightforward. □

Let $D := \{|z| \leq 1\}$, the closed unit disk.

Definition 5. For $w_n = u_n + iv_n \in D$,

$$\Phi_n(w_n; \theta) := \theta - \frac{2}{n} \arctan\left(\frac{u_n \sin(n\theta) + v_n \cos(n\theta)}{1 + u_n \cos(n\theta) + v_n \sin(n\theta)}\right)$$

when $1 + u_n \cos(n\theta) + v_n \sin(n\theta) \neq 0$ and extend the definition to all $\theta \in \mathbb{R}$ by insisting that Φ_n is right continuous. We also define

$$\phi_n(w_n; z) := e^{i\Phi_n(w_n; \theta)}, \quad z = e^{i\theta}$$

and

$$\Sigma_N(w; \theta) = \Phi_N(w_N) \circ \dots \circ \Phi_1(w_1)(\theta)$$

This agrees with our previous definition of $\Phi_n(w_n)$ when $w_n \in \Delta$.

Proposition 7. Suppose $w_n \in D$. (a) $\Phi_n(w_n) \in \widetilde{CDF}(S^1)$ and $\Phi_n(w_n)$ is uniquely determined by the normalized distributional derivative $\frac{1}{2\pi}d\Phi_n \in Prob(S^1)$.

(b) Suppose that $|w_n| = 1$. Then Φ_n has image consisting of the $\frac{1}{n}$ th roots of $1/w_n = w_n^*$, and the points of discontinuity are the $\frac{1}{n}$ th roots of $-1/w_n = -w_n^*$. Thus Φ_n is a step function with the length and height of each step given by $w\pi/n$.

Proof. (a) is clear for $w_n \in \Delta$. It will follow from (b) in the case $w_n \in S^1$.

Suppose that $w_n \in S^1$. Then

$$\phi_n(z)^n = z^n \frac{1 + \bar{w}_n z^{-n}}{1 + w_n z^n} = \frac{1}{w_n}$$

This implies the first half of the first part of (b).

When $w_n \in \Delta$

$$\Phi'_n(\theta) = \frac{1 - |w_n|^2}{|1 + w_n z^n|^2}$$

By letting w_n tend to the circle, we see that the jumps will occur when the denominator tends to zero, which is at the $\frac{1}{n}$ the roots of $-1/w_n$. This completes the proof of (b), and hence also of (a). \square

Question 10. *Determine*

$$w \in \prod_{n=1}^{\infty} D : \left\{ \frac{1}{2\pi} d\Sigma_N(w) \right\} \text{ has a unique limit point } \in \text{Prob}(S^1)\}$$

In particular does this set contain $\prod_{n=1}^{\infty} \Delta$?

Suppose that all of the $w_n \in S^1$. Consider the composition Σ_N . In this case Φ_1 has the unique value $1/w_1$. There is a unique limit point exactly when the sequence of unit complex numbers

$$z_n := \phi_n(\cdot(\Phi_2(w_1^{-1})\cdot)), \quad n = 2, 3, \dots$$

has a unique limit point on the circle. This sequence has the form

$$z_1 = w_1^{-1}, \quad z_2 = \sqrt{1/w_2}, \quad z_3 = (1/w_3)^{1/3}, \dots$$

where the roots depend in some way on the preceding point in the sequence. It seems plausible that the way z_{n+1} is chosen is by taking the root $(w_{n+1})^{-1/n}$ which is closest to z_n ; this still leaves some slight ambiguity, which is resolved by the right continuity requirement. These can be chosen to not have a limit.

7. APPENDIX A: VERBLUNSKY COEFFICIENTS

In this appendix, for the convenience of the reader, we recall some basic facts about Verblunsky coefficients, following [30]. To compare with w coefficients, in this appendix we always assume that the ordering is given by $p = \text{identity}$.

Replacing homeomorphisms by their derivatives, the forward mapping in this paper is given by

$$\prod_{n=1}^{\infty} \Delta \rightarrow \text{Prob}(S^1) : w \rightarrow \mu_w = \text{weak}^* - \lim_{N \rightarrow \infty} \left(\prod_{n=1}^N \frac{1 - |w_n|^2}{|1 + w_n \sigma_{n-1}(z)^n|^2} \right) \frac{d\theta}{2\pi}$$

where it may be necessary to consider a smaller domain to guarantee we have a single-valued map. Let $\text{Prob}'(S^1)$ denote the set of probability measures which are nontrivial, in the sense that their support is not a finite set. Verblunsky coefficients define a bijective correspondence

$$\text{Prob}'(S^1) \rightarrow \prod_{n=0}^{\infty} \Delta : \mu \rightarrow (\alpha_n)$$

where if $p_0 = 1, p_1(z), p_2(z), \dots$ are the monic orthogonal polynomials corresponding to the nontrivial measure μ , then $\alpha_n = -p_{n+1}(0)^*$ (e.g. $p_1(z) = z - \alpha_0^*$, $p_2(z) = z^2 + (-\alpha_0^* + \alpha_0\alpha_1^*)z - \alpha_1^*$, and so on).

Suppose that the generalized Fourier expansion of $\mu \in \text{Prob}(S^1)$ is given by

$$\mu = \lim_{N \rightarrow \infty} (1 + \sum_{n=1}^N (c_n z^n + c_n^* z^{-n})) \frac{d\theta}{2\pi}$$

The numbers c_1, c_2, \dots are (in a vague sense) coordinates for $\text{Prob}(S^1)$, subject to the highly complicated positive definiteness constraints

$$(7.1) \quad \det((c_{i-j})_{1 \leq i, j \leq N}) > 0; \quad N = 1, 2, \dots$$

The Verblunsky coefficients undo these constraints in the following elementary way. There is a (Szego) recursion relation

$$p_{n+1}(z) = zp_n(z) - \alpha_n^* p_n^*(z), \quad n \geq 0$$

By integrating this equation with respect to μ , and using the orthogonality of $p_0 = 1$ and p_{n+1} with respect to μ , this implies

$$\alpha_n^* = \frac{\int zp_n(z) d\mu}{\int z^n p_n^*(z) d\mu}$$

Thus

$$\begin{aligned} \alpha_0^* &= \int z d\mu = c_1^*, \quad p_1 = z - \alpha_0^* \\ \alpha_1^* &= \frac{c_2^* - c_1^* c_1^*}{1 - c_1 c_1^*}, \quad p_1(z) = z \dots \end{aligned}$$

and so on. From this it is evident that each α_n is a rational expression in terms of the Fourier coefficients c_n , and conversely the c_n are polynomials in the α_n and their conjugates; see (3.12) of [30]. Thus at least at a heuristic level (given the recursion relation), it is obvious that the Verblunsky mapping is injective. In addition (although this is not as obvious)

$$\mu = \text{weak}^* - \lim_{N \rightarrow \infty} \frac{\prod_{n=0}^{N-1} (1 - |\alpha_n|^2)}{|p_N(z)|^2} \frac{d\theta}{2\pi}$$

(implying the Verblunsky map is surjective), and the constraints (7.1) are diagonalized:

$$\det((c_{i-j})_{1 \leq i, j \leq N}) = \prod_{n=0}^{N-1} (1 - |\alpha_n|^2)^{N-j}$$

(see (8.1) of [30]). Consider the composition of maps

$$\prod_{n=1}^{\infty} \Delta \rightarrow \text{Prob}(S^1) \rightarrow \prod_{m=0}^{\infty} : w \rightarrow \mu_w \rightarrow \alpha$$

Proposition 8. *The sequence with one nonzero element $(0, \dots, 0, w_n, 0, \dots)$ maps to the one nonzero element sequence $(0, \dots, 0, \alpha_{n-1} = -w_n, 0, \dots)$*

Proof. Suppose that

$$d\mu = \frac{1}{2\pi} \frac{1 - |w_n|^2}{(1 + w_n z^n)(1 + \bar{w}_n \bar{z}^n)} |dz|$$

The first n monic orthogonal polynomials are obviously $1, z, \dots, z^{n-1}$; we need the other polynomials \square

This Proposition suggests that the analytic properties of the two coordinates might be similar, since it implies that the two mappings have the same linearization at zero. It is not easy to calculate this composition for sequences with multiple nonzero terms. It is even quite complicated to calculate that

$$\alpha_0^*(w_1, w_2, 0, \dots) = -\frac{\bar{w}_1 + w_1 \bar{w}_2}{1 + w_1^2 \bar{w}_2}$$

8. APPENDIX B. SMOOTHNESS CONDITIONS FOR HOMEOMORPHISMS OF S^1

For a map $\sigma : S^1 \rightarrow S^1$ which is 1-1 and onto, the inverse is also 1-1 and onto. For such a map, if σ is continuous, then the inverse is also continuous. However given a more general smoothness condition S for self-maps of S^1 , the set of homeomorphisms of S^1 satisfying this condition may or may not form a subgroup. In this appendix we recall important examples. For the purposes of this paper, the most important examples are of groups which arise because they fix some kind of geometric structure; these are discussed in more detail in a subsection.

Let $HomeoS$ denote the set of orientation preserving homeomorphisms of S^1 satisfying smoothness condition S , and $HomeoS \cap S^{-1}$ the set of orientation preserving homeomorphisms of S^1 such that both the homeomorphism and its inverse satisfy smoothness condition S . If $HomeoS$ is a group, then $HomeoS = HomeoS \cap S^{-1}$.

We now list a number of examples. The first four examples involve Holder type conditions:

(1) For $s = 0$ and for $s \geq 1$, $Homeo_{C^s}$ is the topological group of orientation preserving homeomorphisms of S^1 which are C^s , where if $s = k + \alpha$, $k = 0, 1, 2, \dots$ and $0 < \alpha < 1$, this refers to functions with k th derivative which is Holder continuous of order α . Note that for $s \geq 1$ the inverse of such a C^s homeomorphism has the same degree of smoothness because of the inverse function theorem.

(2) For fixed $0 < s < 1$ and $S = C^s$, both $HomeoS$ and $HomeoS \cap S^{-1}$ fail to be groups. To see this define

$$\Sigma(\theta) = \text{sign}(\theta) \pi^{1-s} |\theta|^s; \quad -\pi \leq \theta \leq \pi$$

and extend this to a homeomorphism of \mathbb{R} by requiring $\Sigma(\theta + 2\pi) = \Sigma(\theta) + 2\pi$. Then $\Sigma(\theta)$ and its inverse are Holder continuous of order s . But $\Sigma \circ \Sigma$ is only Holder continuous of order s^2 , and $s^2 < s$. Thus it is difficult to filter homeomorphisms by smoothness in the range $0 < s < 1$.

(3) $\phi \in Homeo(S^1)$ is quasisymmetric if there is a constant M such that

$$1/M \leq \frac{\phi(e^{i(\theta+t)}) - \phi(e^{i\theta})}{\phi(e^{i(\theta)}) - \phi(e^{i(\theta-t)})} \leq M$$

for all θ, t (see chapter 16 of [11] for details). There are other characterizations: ϕ is quasisymmetric if and only if it can be extended to a homeomorphism of the disk which is quasiconformal, if and only if it stabilizes the critical Sobolev class $W^{1/2, L^2}(S^1)$ (this has a subtle formulation, see [21]). The set $Homeo_{QS}$ (which is more commonly denoted $QS(S^1)$) of quasisymmetric homeomorphisms of S^1 is a group, and it is also naturally a Banach manifold, but it is not a topological group. Any quasisymmetric homeomorphism is Holder continuous of order s , where

$s = 1/K$ and the homeomorphism has a K -quasiconformal extension to Δ . So $QS(S^1)$ is somehow spread out over the range $0 < s < 1$.

(4) The set $Homeo_{H \cap H^{-1}}$ of orientation preserving homeomorphisms which together with their inverses satisfy a Holder condition of some positive order is a group. I do not know if there is some reasonable topology for this group.

The conditions which we have considered are summarized as: for $s > 1$

$$(8.1) \quad C^s \subset C^1 \subset QS \subset H \cap H^{-1} \subset C^0$$

There is an obvious transition at $s = 1$.

We next consider Sobolev type conditions.

(4) For $s > 3/2$, $Homeo_{W^s}$, the set of orientation preserving homeomorphisms of S^1 which are $W^s := W^{s,L^2}$ (smooth of order s in the L^2 Sobolev sense), is a topological group. More generally, for a compact d -manifold X , the set of homeomorphisms of X which are smooth of order s in the L^2 Sobolev sense is a topological group, provided $s > 1 + d/2$; see [9].

(5) For $S = W^{1,L^1}$, the set $HomeoS$ of homeomorphisms which are absolutely continuous is NOT a group, because the inverse of such a map is not necessarily absolutely continuous. However $HomeoS \cap S^{-1}$, which we propose to abbreviate to $AC(S^1)$ (for absolutely continuous) is a group. It is equal to the group of homeomorphisms which fix the measure class $[d\theta]$. As a consequence this group acts unitarily on half-densities (of the Lebesgue class) on the circle with finite norm. We will discuss the appropriate topology in the subsection below.

(6) For $s = 3/2$, the critical L^2 Sobolev case, a surprising adjustment in the definition is required (It is likely that the set of homeomorphisms, together with their inverses, which are $W^{3/2}$, in the straightforward sense, is not a group, but I do not know this for certain; see [9], [10] and [32] for relevant information). For $s = 3/2$, define

$$Homeo_{W^{3/2}} = \{\sigma \in Homeo(S^1) : \sigma \text{ and } \sigma^{-1} \text{ are absolutely continuous and } \ln(\Sigma') \in W^{1/2,L^2}\}$$

This is again a topological group (I cannot think of a good abbreviation for this group). The important point is that the $1 + 1/2$ condition does imply that σ is quasisymmetric, and quasisymmetric homeomorphisms stabilize $W^{1/2}$, and hence

$$\ln(\Sigma') \in W^{1/2} \iff \ln(\Sigma' \circ \Sigma) \in W^{1/2}$$

We will discuss this group from a different point of view, which reveals its critical nature, in the subsection below.

Remark 7. Whether there is an analogous result for a general compact d -manifold X , i.e. whether one can make sense of "a group of homeomorphisms of X which are " L^2 Sobolev smooth of degree $1 + d/2$ ", is an intriguing open question.

To summarize the group conditions that are most important for us, analogous to (8.2), there are inclusions

$$(8.2) \quad W^{1+s,L^2} \subset W^{1+1/2,L^2} \subset \subset AC \subset QS \subset C^0$$

where $s > 1/2$. For Sobolev exponents there is an obvious transition at $s = 1/2$, analogous to the transition for Holder exponents at $s = 1$.

The conjectural aspects of factorization which we have tentatively discussed can be displayed in the following way:

$$\begin{array}{ccccccccc}
 \prod \Delta & \leftarrow & ?? & \leftarrow & S^1 \times l^2 & \leftarrow & S^1 \times w^{1/2} & \leftarrow & S^1 \times w^s \\
 \uparrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 Loop^1(\mathbb{C} \setminus \{0\})/\mathbb{R}^* & \leftarrow & QS & \leftarrow & AC & \leftarrow & W^{3/2}Homeo & \leftarrow & W^{1+s}Homeo
 \end{array}$$

(where “ l^2 ” really should be $l^2 \cap \prod \Delta$, and so on).

8.1. Some Topological Details. The group $Diff(S^1)$ acts naturally and unitarily on the Hilbert space of half-densities associated to the Lebesgue class of the manifold S^1 . We identify these densities with functions using the metric $d\theta$,

$$L^2(S^1, d\theta) \rightarrow L^2\Omega^{1/2}(S^1) : f \rightarrow f|d\theta|^{1/2}$$

Then $\phi \in Diff(S^1)$ corresponds to a unitary operator U_ϕ of $L^2(S^1)$, where

$$U_{\phi^{-1}} : f \rightarrow (\Phi')^{1/2} f \circ \phi$$

Proposition 9. *The closure of $Diff(S^1)$ in the strong operator topology for $U(L^2(S^1))$ is*

$$AC(S^1) = \{\phi : \phi \text{ and } \phi^{-1} \text{ are absolutely continuous}\}$$

Thus $U_{\phi_n} \rightarrow U_\phi$ strongly if and only if $\phi_n \rightarrow \phi$ and $\psi_n \rightarrow \psi$ in $W_{L^1}^1$, where $\psi = \phi^{-1}$.

This representation does not have a natural vacuum. But given the metric $d\theta$, we can consider the orbit through $|d\theta|^{1/2}$. This suggests that $AC(S^1)/S^1$ might have a Euclidean description (hence the question we posed in the introduction).

Given a polarized separable Hilbert space, $H = H_+ \oplus H_-$, and a symmetrically normed ideal \mathcal{I} , there is an associated Banach $*$ -algebra, $\mathcal{L}(\mathcal{I})$, which consists of bounded operators on H , represented as two by two matrices with respect to the polarization, such that the norm

$$\left| \begin{pmatrix} A & \\ & D \end{pmatrix} \right|_{\mathcal{L}} + \left| \begin{pmatrix} & B \\ C & \end{pmatrix} \right|_{\mathcal{I}}$$

is finite. The $*$ -operation is the usual adjoint operation. The corresponding unitary group is

$$U_{(\mathcal{I})} = U(H) \cap \mathcal{L}(\mathcal{I})$$

it is referred to as a restricted unitary group in [26]. Geometrically this group is the group of automorphisms of a Grassmannian (Finsler) symmetric space modelled on \mathcal{I} . There are two obvious topologies on $U_{(\mathcal{I})}$. The first is the induced Banach topology, and in this topology $U_{(\mathcal{I})}$ has the additional structure of a Banach Lie group. The second is the Polish topology τ_{KM} for which convergence means that for $g_n, g \in U_{(\mathcal{I})}$, $g_n \rightarrow g$ if and only if $g_n \rightarrow g$ strongly and

$$\begin{pmatrix} & B_n \\ C_n & \end{pmatrix} \rightarrow \begin{pmatrix} & B \\ C & \end{pmatrix} \quad \text{in } \mathcal{I}$$

The following fundamental theorem (with \mathbb{R} in place of S^1) first appeared in full generality in the dissertation of Semmes (Theorem 3 of [29]).

Theorem 8.1. *Consider the Hardy space polarization for $L^2(S^1)$.*

(a) *For $p = \infty$,*

$$AC(S^1) \cap U_{(\mathcal{L}_\infty)} = \{\phi : \log \Phi' \in VMO\}$$

(b) *For $1 \leq p < \infty$,*

$$AC(S^1) \cap U_{(\mathcal{L}_p)} = \{\phi : \log \Phi' \in B^{1/p}\}$$

The circle S^1 has two distinct real spin structures, periodic (or trivial) and antiperiodic (or Mobius). In the latter case there is a natural action by $Diff(S^1)^{(2)}$, the double cover. The complexification of the antiperiodic spin structure is trivial, but not equivariantly trivial. In each case there is a natural Hilbert space structure for half-forms, denoted by H_p and H_a , respectively. We will identify both of these spaces with H as follows. In the periodic case the identification is simply

$$H \rightarrow H_p : f \rightarrow f(d\theta)^{1/2}$$

In the antiperiodic case there is a polarization

$$H_a = H_a^+ \oplus H_a^-$$

where H_a^\pm is the closure of holomorphic sections of the spin bundle for the disk D and D^* , respectively. There is an isomorphism of polarized spaces

$$H \rightarrow H_a : f \rightarrow f(dz)^{1/2}$$

Let

$$AC(S^1)^{(2)} \rightarrow U(H) : \tilde{\phi} \rightarrow V_{\tilde{\phi}} = \begin{pmatrix} A_a(\tilde{\phi}) & B_a(\tilde{\phi}) \\ C_a(\tilde{\phi}) & D_a(\tilde{\phi}) \end{pmatrix}$$

denote the strong operator completion of the induced action on H .

Corollary 2. $W^{1+1/2}Homeo(S^1)$ is the strong operator completion of $Diff(S^1)^{(2)}$ in $U_{(2)}$, i.e.

$$V_{\tilde{\phi}} \in U_{(2)} \Leftrightarrow U_{\phi} \in U_{(2)} \Leftrightarrow \log \Phi' \in W^{1/2, L^2}$$

For other perspectives, see [21] and [32].

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